Ch 10 The z-transform

- 10.1 The *z*-transform (p.741)
  - ➢We will derive the *z*-transform by examining the effect of applying a complex exponential input to an LTI system.
    - ☞The input signal $x[n]=z^n$ is a complex exponential signal.
      - ■ $z=re^{j\omega}$.
    - ☞Impulse response: $h[n]$.
    - ☞System output $y[n]=h[n]*x[n]$

$$y[n]=\sum_{k=-\infty}^{\infty}h[k]x[n-k]=\sum_{k=-\infty}^{\infty}h[k]z^{n-k}=z^n\sum_{k=-\infty}^{\infty}h[k]z^{-k}$$

- ■ $H(z)=\sum_{n=-\infty}^{\infty}h[n]z^{-n}$

- The *z*-transform of a general discrete-time signal *x*[*n*] is define as

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

  - ☞ $x[n] \xleftrightarrow{Z} X(z)$

- The relation between the *z*-transform and DTFT.

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n}$$

  - ☞We can see that $X(re^{j\omega})$ is the Fourier transform of the sequence *x*[*n*] multiplied by real exponential r^{-n}.

$$x[n]r^{-n} \xleftrightarrow{DTFT} X(re^{j\omega})$$

  - ☞r=1, |z|=1, $x[n] \xleftrightarrow{DTFT} X(z)|_{z=e^{j\omega}} = X(e^{j\omega})$

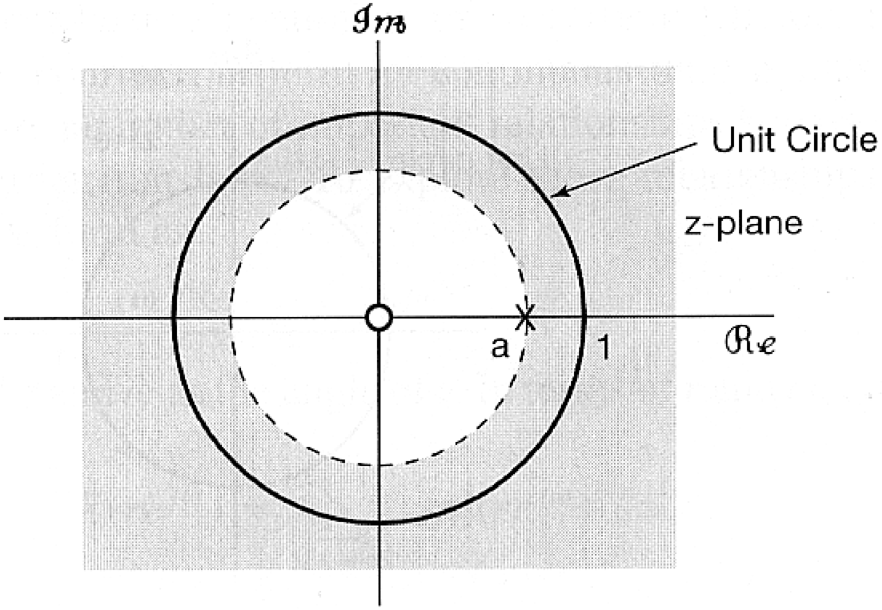
- Ex.10.1 $x[n]=a^n u[n]$. Find the z-transform of $x[n]$.

$$X(z)=\sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty}(az^{-1})^n$$

- ☞ For convergence of $X(z)$, we require that $\sum_{n=0}^{\infty} |az^{-1}|^n < \infty$
  - ■ $|az^{-1}| < 1$ or $|z| > |a|$
- ☞ $X(z) = \sum_{n=0}^{\infty}(az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad |z| > |a|.$
  - ■ for a=1,

$$X(z) = \frac{1}{1-z^{-1}}, \qquad |z| > 1.$$

  - ■ 0<a<1



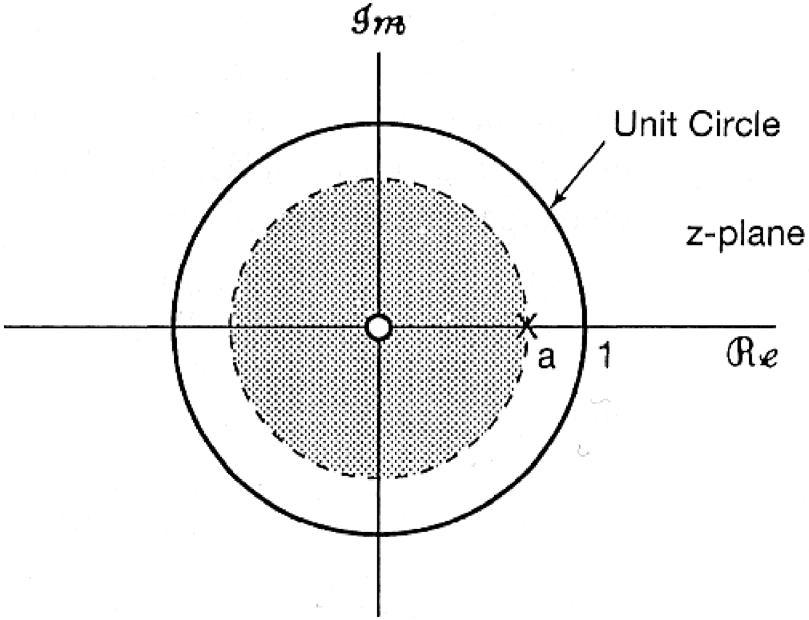
- Ex. 10.2 $x[n]=-a^n u[-n-1]$. Find the z-transform of $x[n]$.

$$X(z) = -\sum_{n=-\infty}^{+\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n.$$

☞ $$X(z) = 1 - \frac{1}{1-a^{-1}z} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad |z| < |a|.$$

- ■ $0<a<1$



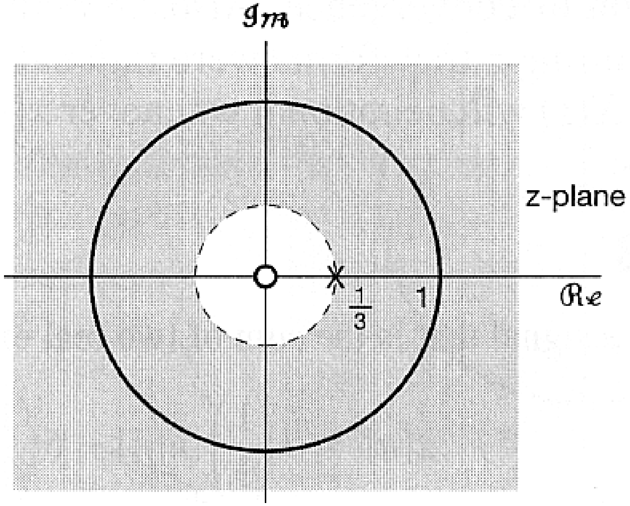
- Ex.10.3 $x[n]=7(\frac{1}{3})^n u[n]-6(\frac{1}{2})^n u[n]$. Find the *z*-transform of $x[n]$.

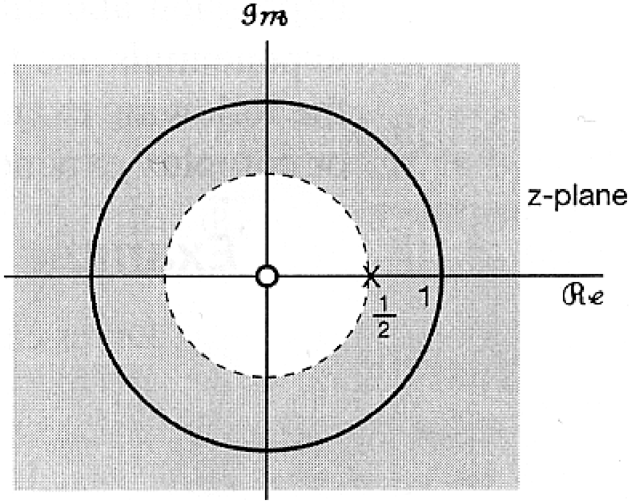
$$
\begin{aligned}
X(z) &= \sum_{n=-\infty}^{+\infty}\left\{7\left(\frac{1}{3}\right)^n u[n]-6\left(\frac{1}{2}\right)^n u[n]\right\}z^{-n}\\
&= 7\sum_{n=-\infty}^{+\infty}\left(\frac{1}{3}\right)^n u[n]z^{-n}-6\sum_{n=-\infty}^{+\infty}\left(\frac{1}{2}\right)^n u[n]z^{-n}\\
&= 7\sum_{n=0}^{\infty}\left(\frac{1}{3}z^{-1}\right)^n-6\sum_{n=0}^{+\infty}\left(\frac{1}{2}z^{-1}\right)^n\\
&= \frac{7}{1-\frac{1}{3}z^{-1}}-\frac{6}{1-\frac{1}{2}z^{-1}}=\frac{1-\frac{3}{2}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z^{-1})}\\
&= \frac{z(z-\frac{3}{2})}{(z-\frac{1}{3})(z-\frac{1}{2})}.
\end{aligned}
$$

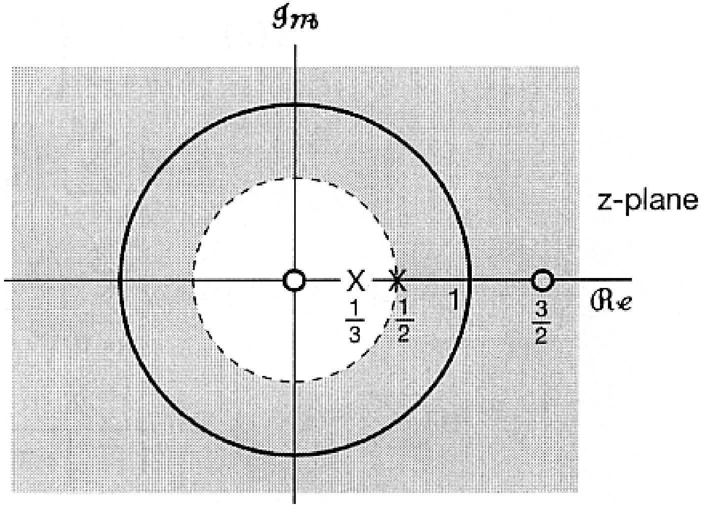
$$\left(\frac{1}{3}\right)^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1-\frac{1}{3}z^{-1}}, \quad |z| > \tfrac{1}{3}$$

$$\left(\frac{1}{2}\right)^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \tfrac{1}{2},$$

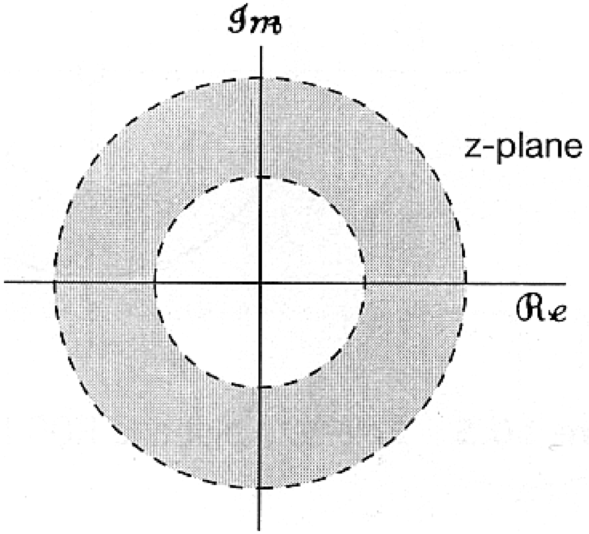
$$7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n] \overset{z}{\longleftrightarrow} \frac{7}{1-\frac{1}{3}z^{-1}} - \frac{6}{1-\frac{1}{2}z^{-1}}, \quad |z| > \tfrac{1}{2},$$







- 10.2 The ROC for the *z*-transform (p.748)
  - **Property 1**: The ROC of *X*(*z*) consists of a ring in the *z*-plane centered about the origin.
    - ☞ $$\sum_{n=-\infty}^{+\infty} |x[n]| r^{-n} < \infty.$$
    - ☞ ROC as a ring in the *z*-plane. In some cases, the inner boundary can extend inward to the origin, in which case the ROC becomes a disk. In other cases, the outer boundary can extend outward to infinity.



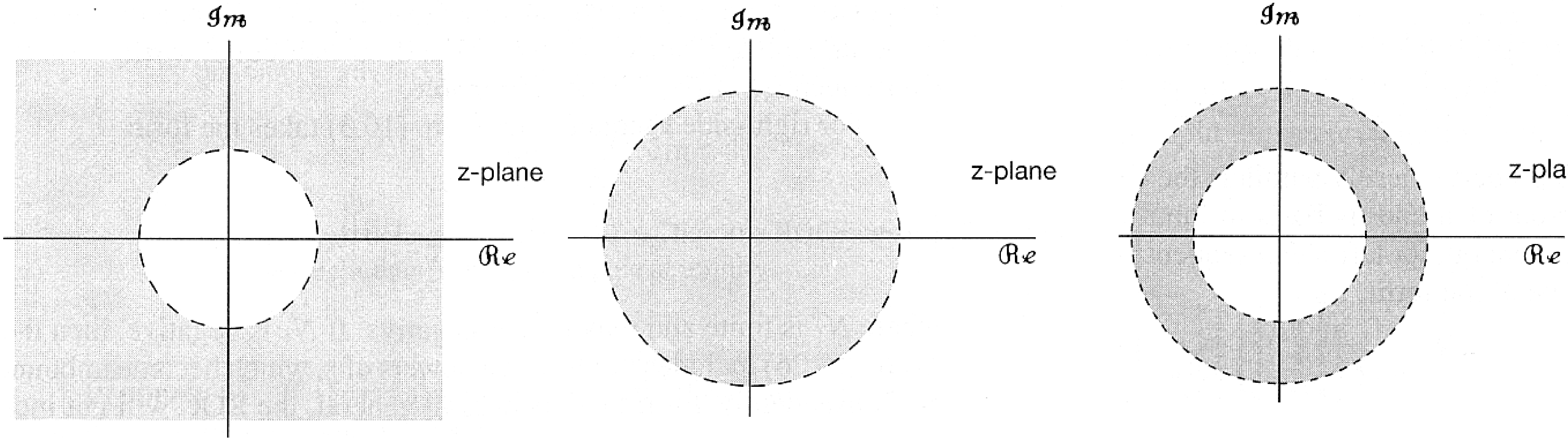
- **Property 2**: The ROC does not contain any poles.

- **Property 3**: If $x[n]$ is of finite duration, then the ROC is the entire z-plane, except possibly $z=0$ and/or $z=\infty$.
  - ☞ Ex. $x[-1]=1$, $x[0]=2$, $x[1]=3$.
  - ☞ $\delta[n]$ ---> ?
  - ☞ $\delta[n-1]$ ---> ?

- **Property 4**: If $x[n]$ is a right-sided sequence, and if the circle $|z|=r_0$ is in the ROC, then all finite values of z for which $|z|>r_0$ will also be in the ROC. (p.751)

- **Property 5**: If $x[n]$ is a left-sided sequence, and if the circle $|z|=r_0$ is in the ROC, then all finite values of z for which $0<|z|<r_0$ will also be in the ROC.

- **Property 6**: If $x[n]$ is two sided, and if the circle $|z|=r_0$ is in the ROC, then the ROC will consist of a ring in the z-plane that includes the circle $|z|=r_0$.



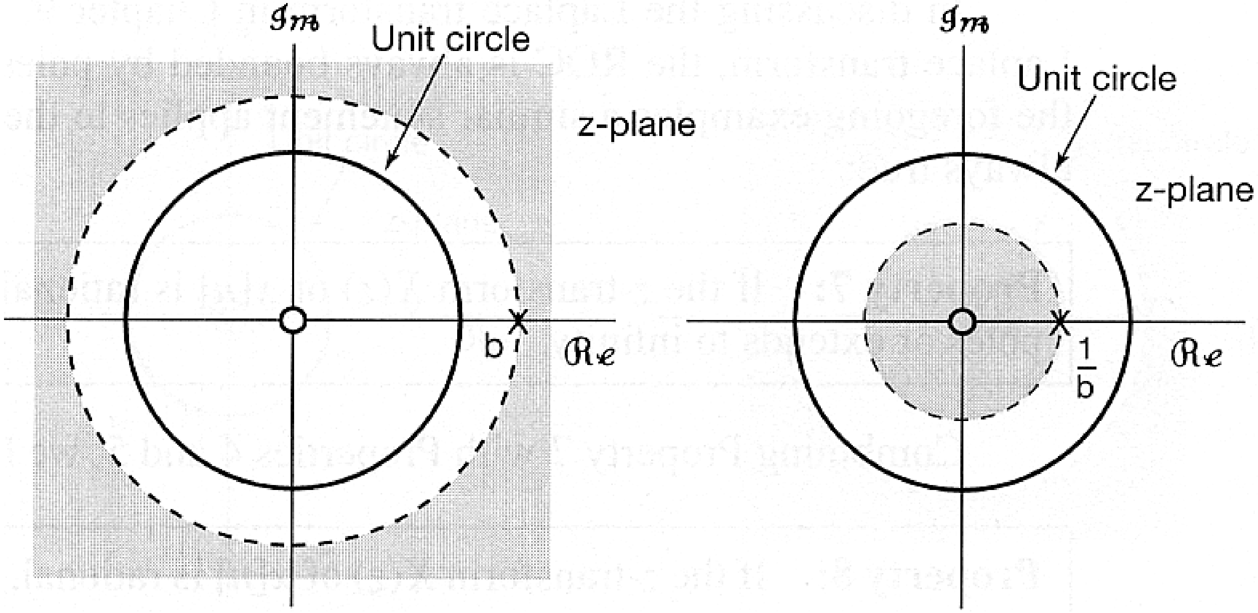
- ➢Ex.10.7 $x[n]=b^{|n|}$, $b>0$. Find the z-transform of $x[n]$.

☞ $x[n]=b^{n}u[n]+b^{-n}u[-n-1]$

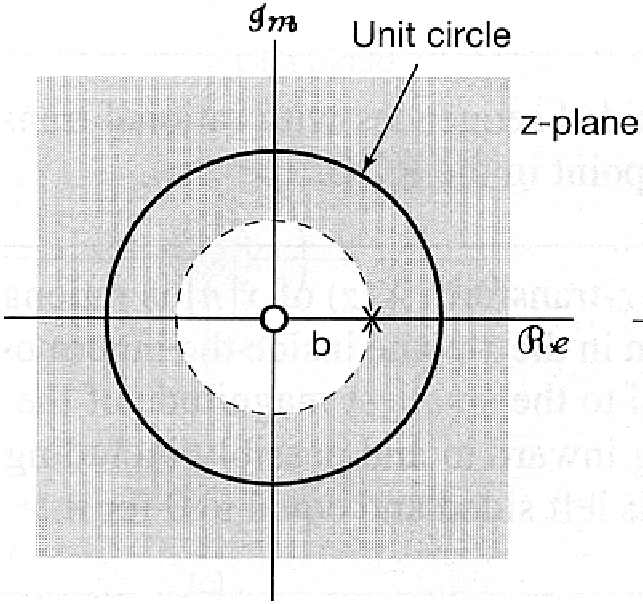
☞ $$b^{n}u[n]\xleftrightarrow{Z}\frac{1}{1-bz^{-1}},\quad |z|>b$$

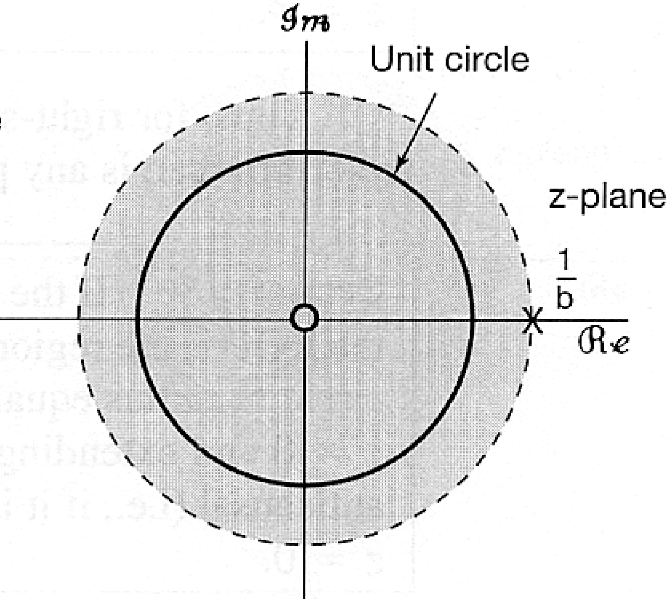
☞ $$b^{-n}u[-n-1]\xleftrightarrow{Z}\frac{-1}{1-b^{-1}z^{-1}},\quad |z|<\frac{1}{b}$$

① b>1

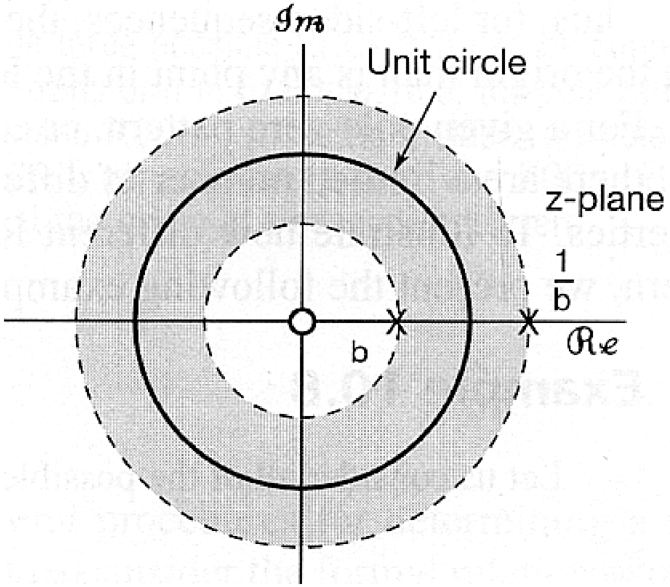


② 0<b<1





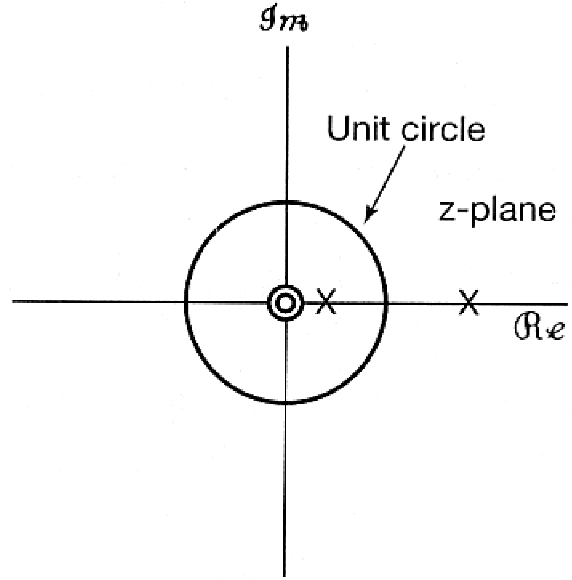
$$X(z)=\frac{1}{1-bz^{-1}}-\frac{1}{1-b^{-1}z^{-1}},\quad b<|z|<\frac{1}{b}$$

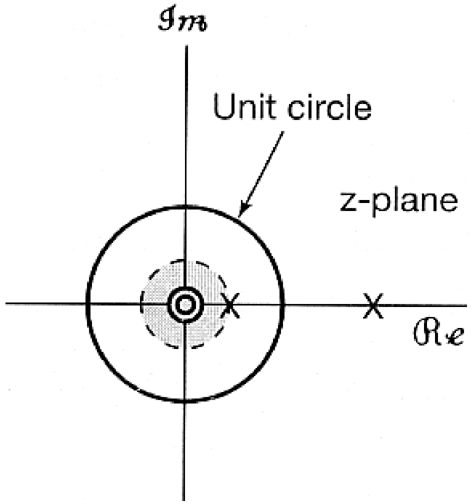


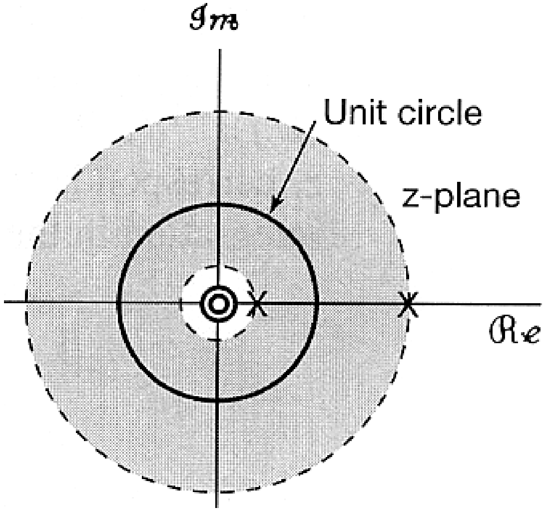
- **Property 7**: If the z-transform $X(z)$ of $x[n]$ is rational, then its ROC is bounded by poles or extends to infinity.
  - ☞Combining property 7 with properties 4 and 5, we have
- **Property 8**: If the z-transform $X(z)$ of $x[n]$ is rational, and if $x[n]$ is right sided, then the ROC is the region in the z-plane outside the outermost pole—i.e., outside the circle of radius equal to the largest magnitude of the poles of $X(z)$. Furthermore, if $x[n]$ is causal, then the ROC also includes $z= \infty$.
- **Property 9**: If the z-transform $X(z)$ of $x[n]$ is rational, and if $x[n]$ is left sided, then the ROC is the region in the z-plane inside the innermost nonzero pole—i.e., inside the circle of radius equal to the smallest magnitude of the poles of $X(z)$ other than any at $z=0$ and extending inward to and possible including $z=0$. In particular, if $x[n]$ is anticausal, then the ROC also includes $z=0$.

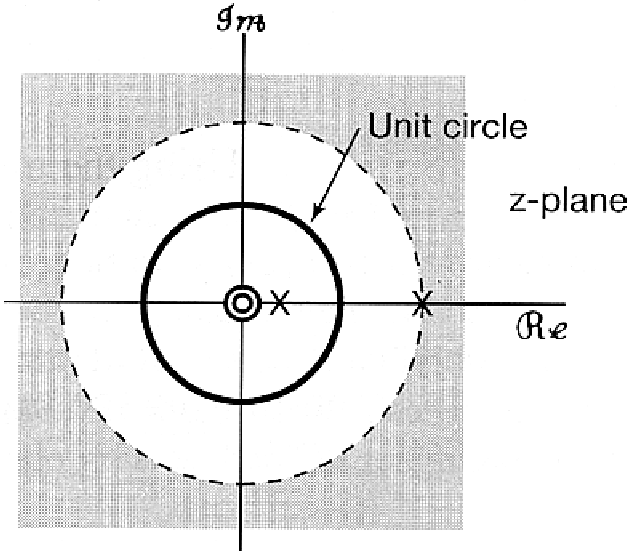
- Ex. 10.8 Let us consider all of the possible ROCs that can be connected with the function

$$X(z) = \frac{1}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})}$$









- 10.3 The inverse z-transform
  - $x[n]r^{-n} \xleftrightarrow{DTFT} X(re^{j\omega})$,where $x[n]r^{-n} = \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})e^{j\omega n}d\omega$

    $$x[n] = \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})(re^{j\omega})^n d\omega$$

  - $x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz,$

  - Ex.10.9 Find the inverse z-transform for $X(z)$

    $$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})}, \quad |z| > \frac{1}{3}.$$

- ☞ $$X(z)=\frac{1}{1-\frac{1}{4}z^{-1}}+\frac{2}{1-\frac{1}{3}z^{-1}}$$

- ☞ $x[n]=x_1[n]+x_2[n]$

$$x_1[n]\xleftrightarrow{Z}\frac{1}{1-\frac{1}{4}z^{-1}},\quad |z|>\frac{1}{4} \qquad x_1[n]=(\frac{1}{4})^n u[n]$$

$$x_2[n]\xleftrightarrow{Z}\frac{2}{1-\frac{1}{3}z^{-1}},\quad |z|>\frac{1}{3} \qquad x_2[n]=2(\frac{1}{3})^n u[n]$$

- ☞ $$x[n]=(\frac{1}{4})^n u[n]+2(\frac{1}{3})^n u[n]$$

- Ex. $X(z) = \dfrac{3 - \frac{5}{6}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})}, \quad \dfrac{1}{4} < |z| < \dfrac{1}{3}$

☞ $x[n] = (\frac{1}{4})^n u[n] - 2(\frac{1}{3})^n u[-n-1]$

- Ex. $X(z) = \dfrac{3 - \frac{5}{6}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})}, \quad |z| < \dfrac{1}{4}\ .$

☞ $x[n] = -(\frac{1}{4})^n u[-n-1] - 2(\frac{1}{3})^n u[-n-1]$

- Ex. 10.12 $X(z) = 4z^2 + 2 + 3z^{-1}, \quad 0 < |z| < \infty$

$$x[n] = 4\delta[n+2] + 2\delta[n] + 3\delta[n-1] \qquad x[n] = \begin{cases} 4, & n = -2 \\ 2, & n = 0 \\ 3, & n = 1 \\ 0, & \text{othewise} \end{cases}$$

➢Ex. 10.13 Find the inverse z-transform for $X(z)$

$$X(z) = \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

$$\begin{array}{rl} & 1 + az^{-1} + a^2z^{-2} + \cdots \\ \cline{2-2} 1 - az^{-1} \big) & 1 \\ & \underline{1 - az^{-1}} \\ & \quad az^{-1} \\ & \quad \underline{az^{-1} - a^2z^{-2}} \\ & \qquad\quad a^2z^{-2} \end{array}$$

$$\frac{1}{1 - az^{-1}} = 1 + az^{-1} + a^2z^{-2} + \ldots.$$

$$X(z) = \frac{1}{1 - az^{-1}}, \quad |z| < |a|$$

$$\begin{array}{rl} & -a^{-1}z - a^{-2}z^2 - \cdots \\ \cline{2-2} -az^{-1} + 1 \big) & 1 \\ & \underline{1 - a^{-1}z} \quad , \\ & \qquad a^{-1}z \end{array}$$

$$\frac{1}{1 - az^{-1}} = -a^{-1}z - a^{-2}z^2 - \ldots.$$

➢Ex. Find the inverse z-transform for $H(z)$

$$H\left(z\right)=\frac{z^2(z-\frac{1}{2})}{(z-\frac{2}{3})(z-\frac{1}{3})(z-\frac{1}{4})}=\frac{z^3-\frac{z^2}{2}}{z^3-\frac{15}{12}z^2+\frac{17}{36}z-\frac{1}{18}}$$

$$\begin{array}{r l} & 1+\frac{3}{4}z^{-1}+\frac{67}{144}z^{-2}+\cdots \\ z^3-\frac{15}{12}z^2+\frac{17}{36}z-\frac{1}{18}\Big) & \overline{z^3-\frac{z^2}{2}\qquad\qquad\qquad\qquad\qquad} \\ & \underline{z^3-\frac{15}{12}z^2+\frac{17}{36}z-\frac{1}{18}} \\ & \frac{3}{4}z^2-\frac{17}{36}z+\frac{1}{18} \\ & \underline{\frac{3}{4}z^2-\frac{45}{48}z+\frac{51}{144}-\frac{3}{72}z^{-1}} \\ & \frac{67}{144}z\cdots \\ & \vdots\ \ \vdots \end{array}$$

$$h[n]=1+\frac{3}{4}z^{-1}+\frac{67}{144}z^{-2}+\cdots$$

☞ ROC: $|z|>\frac{2}{3}$

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1 - z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1 - z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z, except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $\alpha^n u[n]$	$\frac{1}{1 - \alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
6. $-\alpha^n u[-n-1]$	$\frac{1}{1 - \alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
7. $n\alpha^n u[n]$	$\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
8. $-n\alpha^n u[-n-1]$	$\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
9. $[\cos \omega_0 n]u[n]$	$\frac{1 - [\cos \omega_0]z^{-1}}{1 - [2\cos \omega_0]z^{-1} + z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin \omega_0 n]u[n]$	$\frac{[\sin \omega_0]z^{-1}}{1 - [2\cos \omega_0]z^{-1} + z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n \cos \omega_0 n]u[n]$	$\frac{1 - [r\cos \omega_0]z^{-1}}{1 - [2r\cos \omega_0]z^{-1} + r^2 z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n \sin \omega_0 n]u[n]$	$\frac{[r\sin \omega_0]z^{-1}}{1 - [2r\cos \omega_0]z^{-1} + r^2 z^{-2}}$	$\lvert z\rvert > r$

- 10.5 Properties of the z-transform (p.767)
  - Linearity

$$x_1[n] \overset{z}{\longleftrightarrow} X_1(z), \quad \text{with ROC} = R_1,$$

$$x_2[n] \overset{z}{\longleftrightarrow} X_2(z), \quad \text{with ROC} = R_2,$$

$$ax_1[n] + bx_2[n] \overset{z}{\longleftrightarrow} aX_1(z) + bX_2(z), \quad \text{with ROC containing } R_1 \cap R_2.$$

  - Time shifting

$$x[n] \overset{Z}{\longleftrightarrow} X(z), \text{ with ROC=}R$$

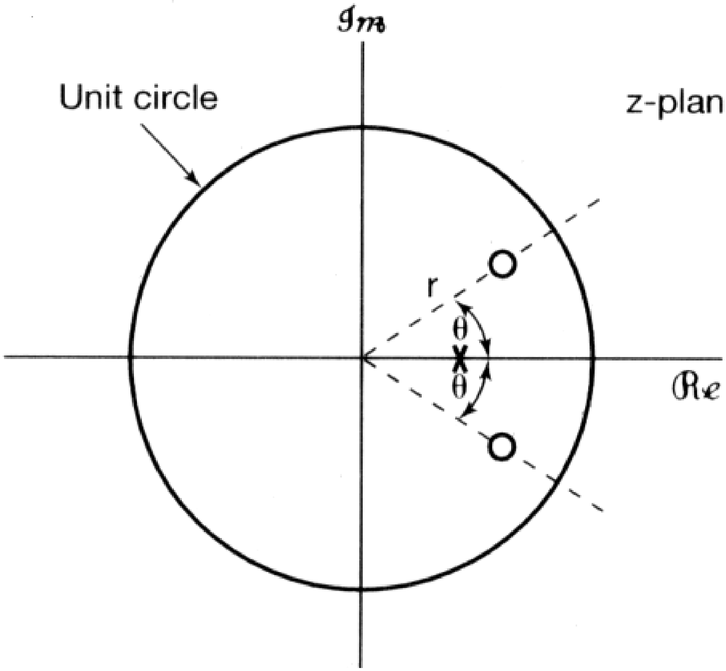
$$x[n - n_0] \overset{z}{\longleftrightarrow} z^{-n_0}X(z), \quad \text{with ROC} = R\text{, except for the possible addition or deletion of the origin or infinity.}$$

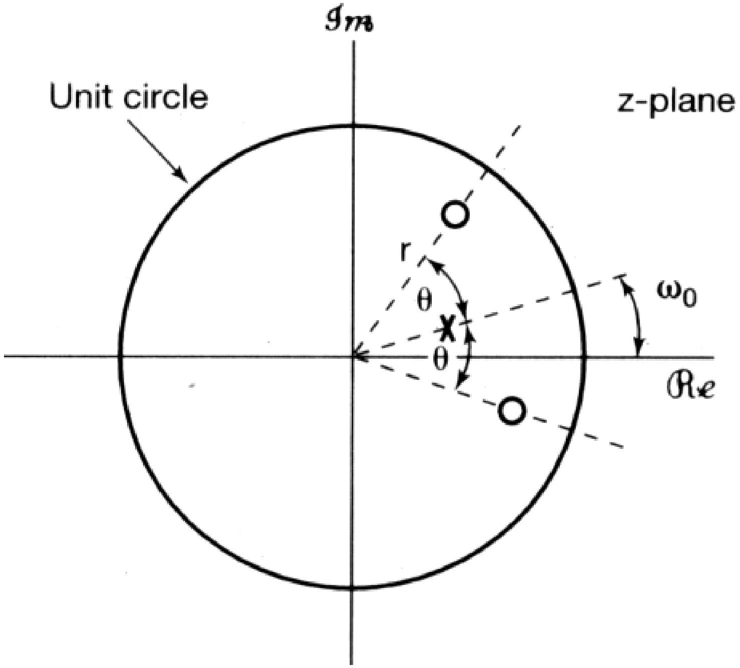
- Scaling in the z-transform

$$x[n] \overset{Z}{\longleftrightarrow} X(z), \text{ with ROC=}R$$

$$z_0^n x[n] \overset{Z}{\longleftrightarrow} X\left(\frac{z}{z_0}\right), \text{ with ROC } = |z_0|R,$$

$$e^{j\omega_0 n} x[n] \overset{Z}{\longleftrightarrow} X(e^{-j\omega_0} z).$$





➢Time reversal

$$x[n] \xleftrightarrow{Z} X(z),\ \text{with ROC}=R$$

$$x[-n] \xleftrightarrow{Z} X(\tfrac{1}{z}), \quad \text{with ROC} = \frac{1}{R}.$$

Ex. $x[n]=a^n u[n]$.

$$X(z)=\frac{1}{1-az^{-1}},\ |a|<|z|$$

☞ $x[-n]=a^{-n}u[-n]$

$$X(z)=\frac{1}{1-az},\ |z|<|\frac{1}{a}|$$

➢Time expansion

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k \end{cases}$$

☞ has *k*-1 zeros inserted between successive values of the original signal.

☞ $x[n] \xleftrightarrow{Z} X(z),$ with ROC=*R*

$$x_{(k)}[n] \xleftrightarrow{Z} X(z^k), \quad \text{with ROC} = R^{1/k}.$$

☞ $X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n},$

$$\sum_{n=-\infty}^{\infty} x_{(k)}[n]z^{-n} = \sum_{n=-\infty}^{\infty} x[n]z^{-kn} = \sum_{n=-\infty}^{\infty} x[n](z^k)^{-n} = X(z^k)$$

- Conjugation

$$x[n] \xleftrightarrow{z} X(z),\ \text{with ROC}=R$$

$$x^*[n] \xleftrightarrow{z} X^*(z^*), \quad \text{with ROC} = R.$$

☞ If $x[n]$ is real, we can conclude $X(z)=X^*(z^*)$

- ■Thus, if $X(z)$ has a pole (or zero) at $z=z_0$, it must also have a pole (or zero) at the complex conjugate point $z = z_0^*$

- The convolution property

$$x_1[n] \xleftrightarrow{z} X_1(z), \quad \text{with ROC} = R_1,$$

$$x_2[n] \xleftrightarrow{z} X_2(z), \quad \text{with ROC} = R_2,$$

$$x_1[n] * x_2[n] \xleftrightarrow{z} X_1(z)X_2(z), \quad \text{with ROC containing } R_1 \cap R_2.$$

Ex. 10.15 Consider an LTI system for which $y[n]=h[n]*x[n]$, where $h[n]=\delta[n]-\delta[n-1]$.

☞ $h[n]=\delta[n]-\delta[n-1]\xleftrightarrow{Z}H(z)=1-z^{-1}$

■ with ROC equal to the entire z-plane except the origin.

☞ $x[n]\xleftrightarrow{Z}X(z),$ with ROC=R

☞ $Y(z)=H(z)X(z)=(1-z^{-1})X(z)$

■with ROC equal to R, with the possible deletion of $z=0$ and/or addition of $z=1$.

☞Note $y[n]=h[n]*x[n]=x[n]-x[n-1]$

➢Differentiation in the *z*-domain

$$x[n] \xleftrightarrow{Z} X(z), \text{ with ROC=}R$$

$$nx[n] \xleftrightarrow{Z} -z\frac{dX(z)}{dz}, \text{ with ROC=}R$$

Ex. 10.17 *X*(*z*)=ln(1+az^{-1}), |z|>|*a*|, find inverse *z*-transform for *X*(*z*).

☞ $nx[n] \xleftrightarrow{Z} -z\frac{dX(z)}{dz} = \frac{az^{-1}}{1+az^{-1}}, \ |z|>|a|$

☞ $a(-a)^n u[n] \xleftrightarrow{z} \frac{a}{1+az^{-1}}, \quad |z| > |a|.$

☞ $a(-a)^{n-1} u[n-1] \xleftrightarrow{z} \frac{az^{-1}}{1+az^{-1}}, \quad |z| > |a|.$

☞ $x[n] = \frac{-(-a)^n}{n} u[n-1].$

➢The initial-value theorem

☞If $x[n]=0$, $n<0$, then $x[0] = \lim_{z\to\infty} X(z).$

☞ $X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}.$

➢ Ex.10.19 $X(z)=\dfrac{1-\frac{3}{2}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z^{-1})}$, $|z|>1/2$. Find $x[0]$

$$x[0]=\lim_{z\to\infty} X(z)=\frac{1-\frac{3}{2}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z^{-1})}=1$$

TABLE 10.1 PROPERTIES OF THE *z*-TRANSFORM

Section	Property	Signal	z-Transform	ROC
		$x[n]$ $x_1[n]$ $x_2[n]$	$X(z)$ $X_1(z)$ $X_2(z)$	R R_1 R_2
10.5.1	Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2
10.5.2	Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R, except for the possible addition or deletion of the origin
10.5.3	Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R
		$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	z_0R
		$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e., $\|a\|R$ = the set of points $\{\|a\|z\}$ for z in R)
10.5.4	Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1}, where z is in R)
10.5.5	Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in R)
10.5.6	Conjugation	$x^*[n]$	$X^*(z^*)$	R
10.5.7	Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of R_1 and R_2
10.5.7	First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $\|z\| > 0$
10.5.7	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - z^{-1}}X(z)$	At least the intersection of R and $\|z\| > 1$
10.5.8	Differentiation in the z-domain	$nx[n]$	$-z\frac{dX(z)}{dz}$	R

10.5.9

Initial Value Theorem
If $x[n] = 0$ for $n < 0$, then
$$x[0] = \lim_{z\to\infty} X(z)$$

- 10.7 Analysis and Characterization of LTI systems using *z*-transform
  - ➢LTI system
    - ☞$Y(z)=H(z)X(z)$, where $X(z)$, $Y(z)$ and $H(z)$ are the z-transform of the system input, output, and impulse response respectively.
      - ■ $H(z)$ referred to as the system function or transfer function of the system.
  - ➢Causality
    - ☞A causal LTI system has an impulse response $h[n]$ that is zero for $n<0$, and therefore is right-sided.
    - ☞A discrete-time LTI system is causal if and only if the ROC of its system function is the exterior of a circle, include infinity.

☞A discrete-time LTI system with rational system function *H*(*z*) is causal if and only if:

(a) the ROC is the exterior of a circle outside the outermost pole; and

(b) with *H*(*z*) expressed as a ratio of polynomials in *z*, the order of the numerator cannot be greater than the order of the denominator.

- Ex. 10.20 Consider a system with system function whose algebraic expression is

$$H(z)=\frac{z^3-2z^2+z}{z^2+\frac{1}{4}z+\frac{1}{8}}$$

Causal system?

- Stability
  - ☞The stability of a discrete-time LTI system is equivalent to its impulse response being absolutely summable.
    - ■In this case the Fourier transform of $h[n]$ converges, and consequently, the ROC of $H(z)$ must include the unit circle.
  - ☞An LTI system is stable if and only if the ROC of its system function $H(z)$ includes the unit circle, $|z|=1$.
  - ☞A causal LTI system with rational system function $H(z)$ is stable if and only if all of the poles of $H(z)$ lie inside the unit circle—i.e., they must all have magnitude smaller than 1.

Ex. 10.23 Consider a causal system with system function

$$H(z)=\frac{1}{1-az^{-1}}$$

- ☞which has a pole at $z=a$.
- ☞For this system to be stable, its pole must be inside the unit circle.
- ☞ $|a|<1$

➢10.7.3 LTI systems characterized by linear constant-coefficient difference equations

- ☞Ex. 10.25 $y[n]-\frac{1}{2}y[n-1]=x[n]+\frac{1}{3}x[n-1]$

☞ $$Y(z)-\frac{1}{2}z^{-1}Y(z)=X(z)+\frac{1}{3}z^{-1}X(z)$$

☞ $$H(z)=Y(z)/X(z)=\frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

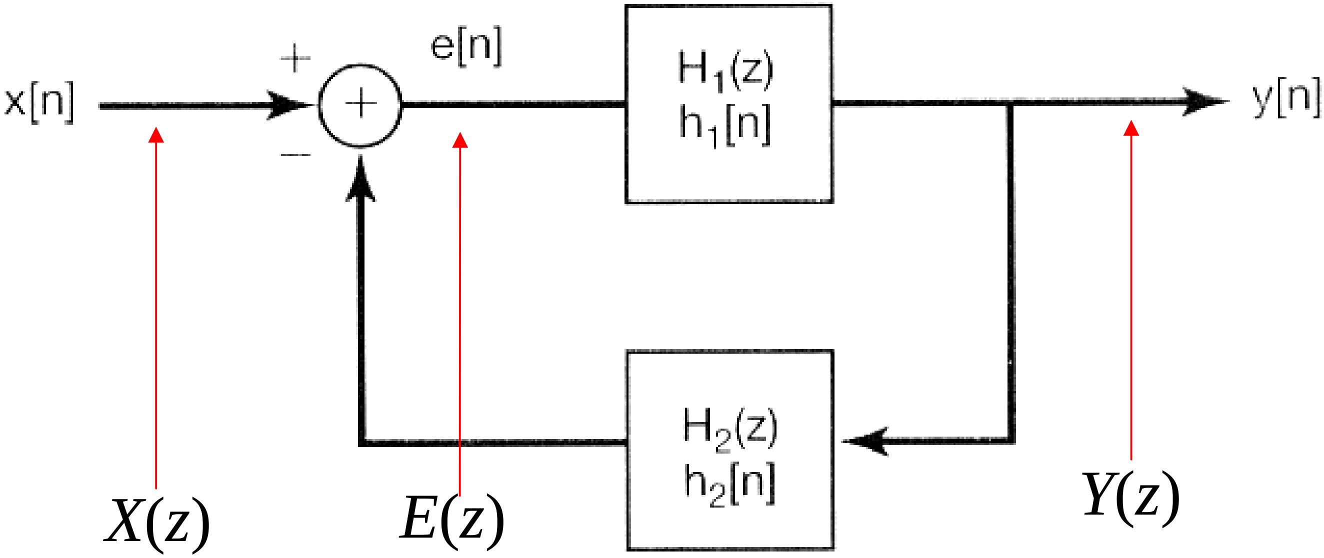
☞ $|z|>0.5$ → $h[n]=(\frac{1}{2})^{n}u[n]+\frac{1}{3}(\frac{1}{2})^{n-1}u[n-1]$

☞ $|z|<0.5$ → $h[n]=-(\frac{1}{2})^{n}u[-n-1]-\frac{1}{3}(\frac{1}{2})^{n-1}u[-n]$

- $$\sum_{k=0}^{N}a_k y[n-k]=\sum_{k=0}^{M}b_k x[n-k]\overset{Z}{\leftrightarrow}\sum_{k=0}^{N}a_k z^{-k}Y(z)=\sum_{k=0}^{M}b_k z^{-k}X(z)$$

☞ $$H(z)=\frac{Y(z)}{X(z)}=\frac{\sum_{k=0}^{M}b_k z^{-k}}{\sum_{k=0}^{N}a_k z^{-k}}$$

- 10.8 System function for interconnections of LTI systems



- $E(z) = X(z) - Y(z)H_2(z)$ and $Y(z) = E(z)H_1(z)$

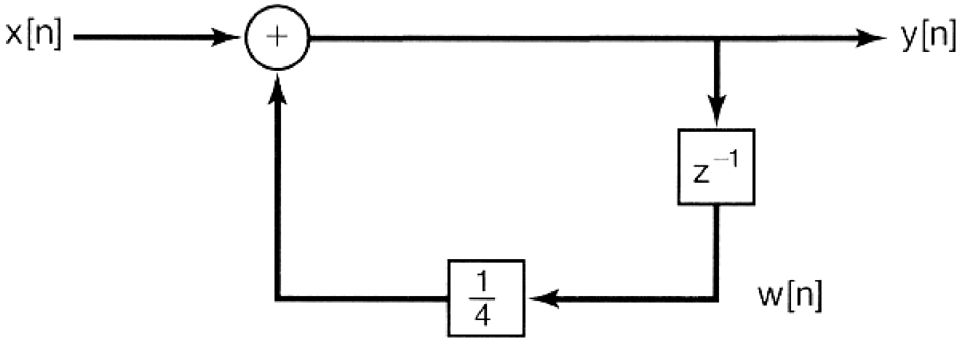
$$Y(z)/H_1(z) = X(z) - Y(z)H_2(z) \rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{H_1(z)}{1 + H_1(z)H_2(z)}$$

Ex. 10.28 Consider the causal LTI system with system function

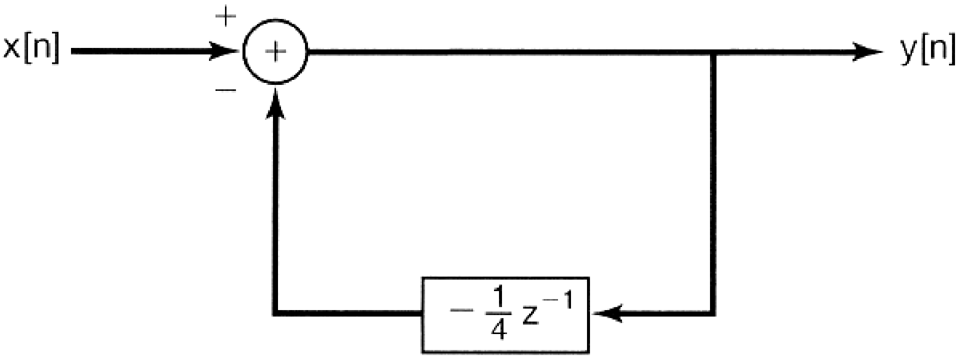
$$H(z)=\frac{1}{1-\frac{1}{4}z^{-1}}$$

☞ $\frac{Y(z)}{X(z)}=\frac{1}{1-\frac{1}{4}z^{-1}}$ ⇨ $Y(z)-\frac{1}{4}Y(z)z^{-1}=X(z)$

☞ $y[n]-0.25y[n-1]=x[n]$



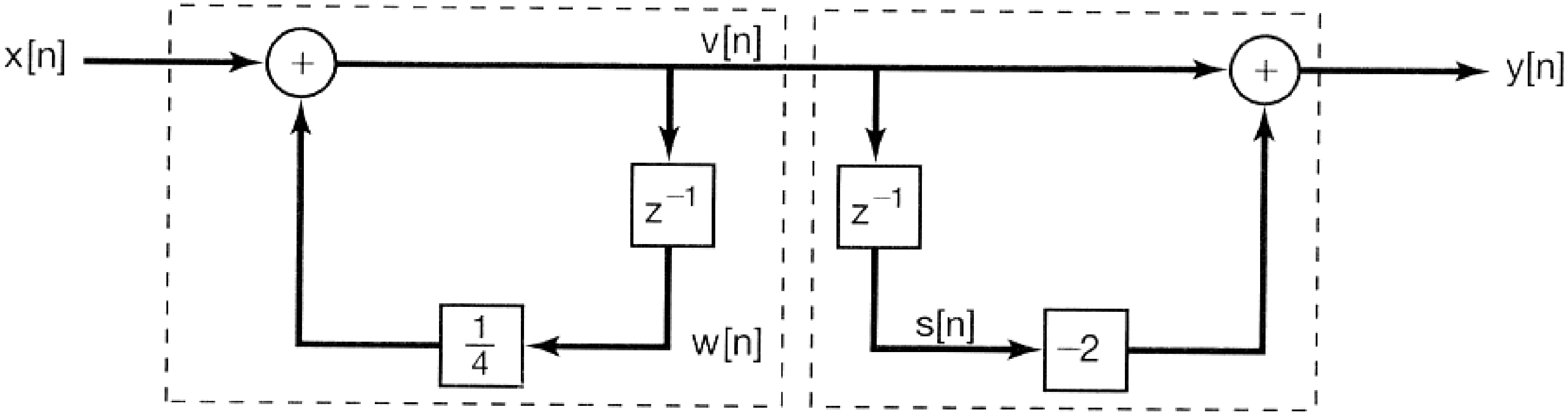
(a)

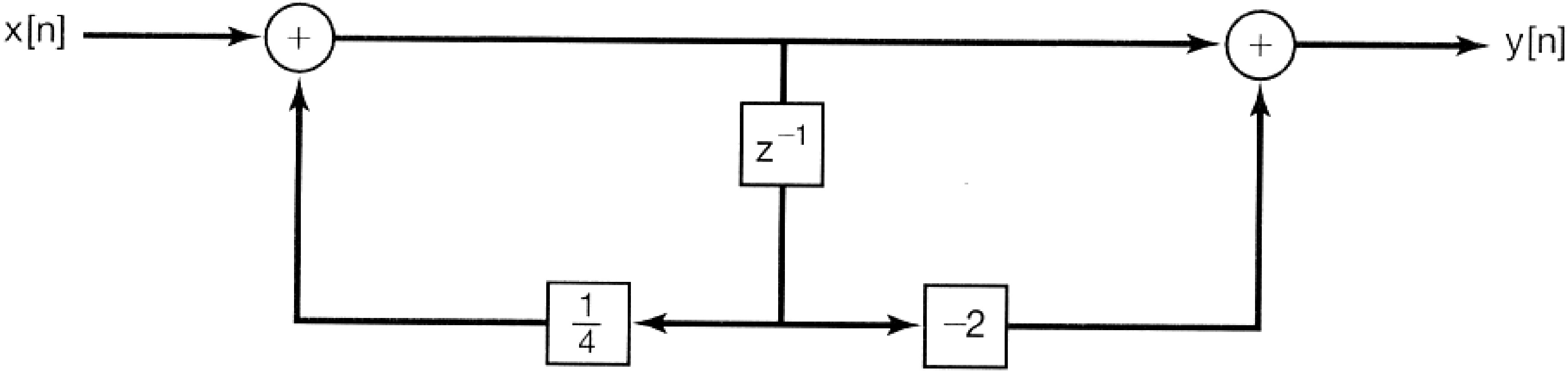


(b)

Ex. 10.29 Consider the causal LTI system with system function

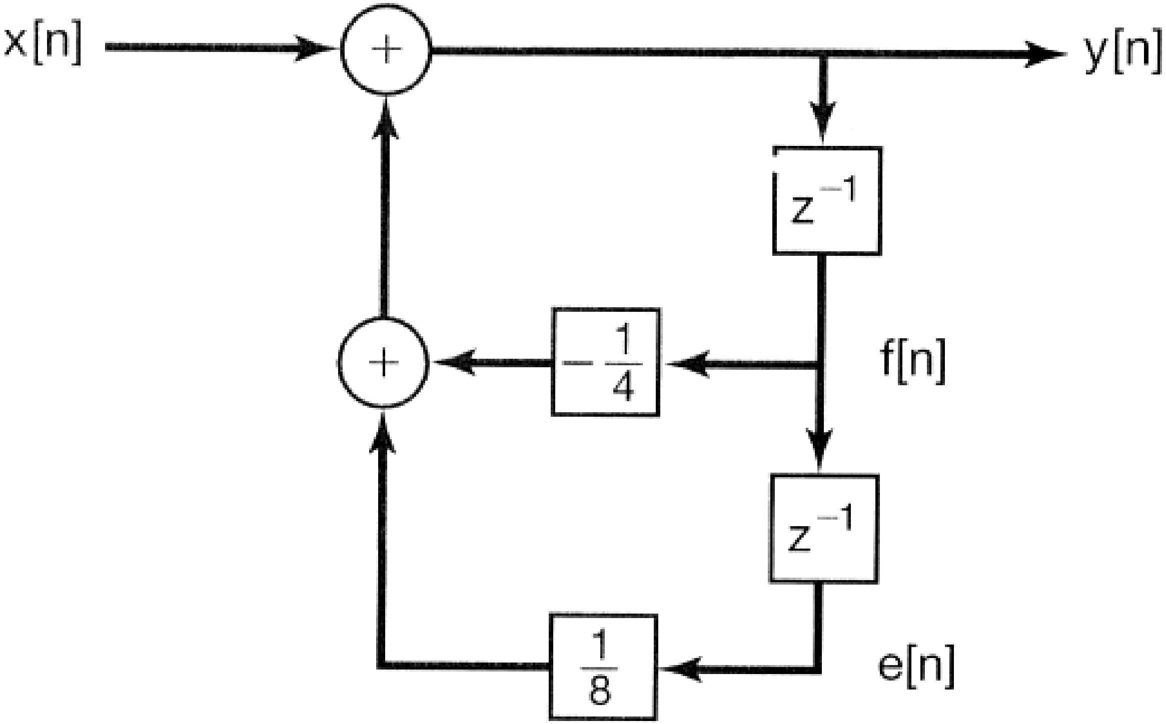
$$H(z)=\frac{1-2z^{-1}}{1-\frac{1}{4}z^{-1}}=\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)(1-2z^{-1})$$

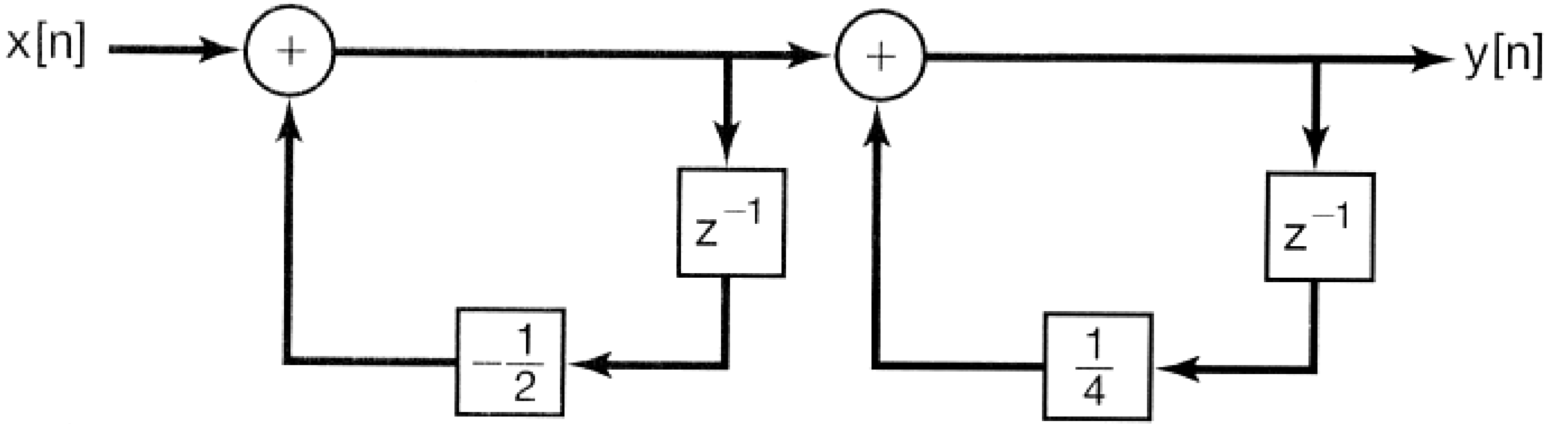


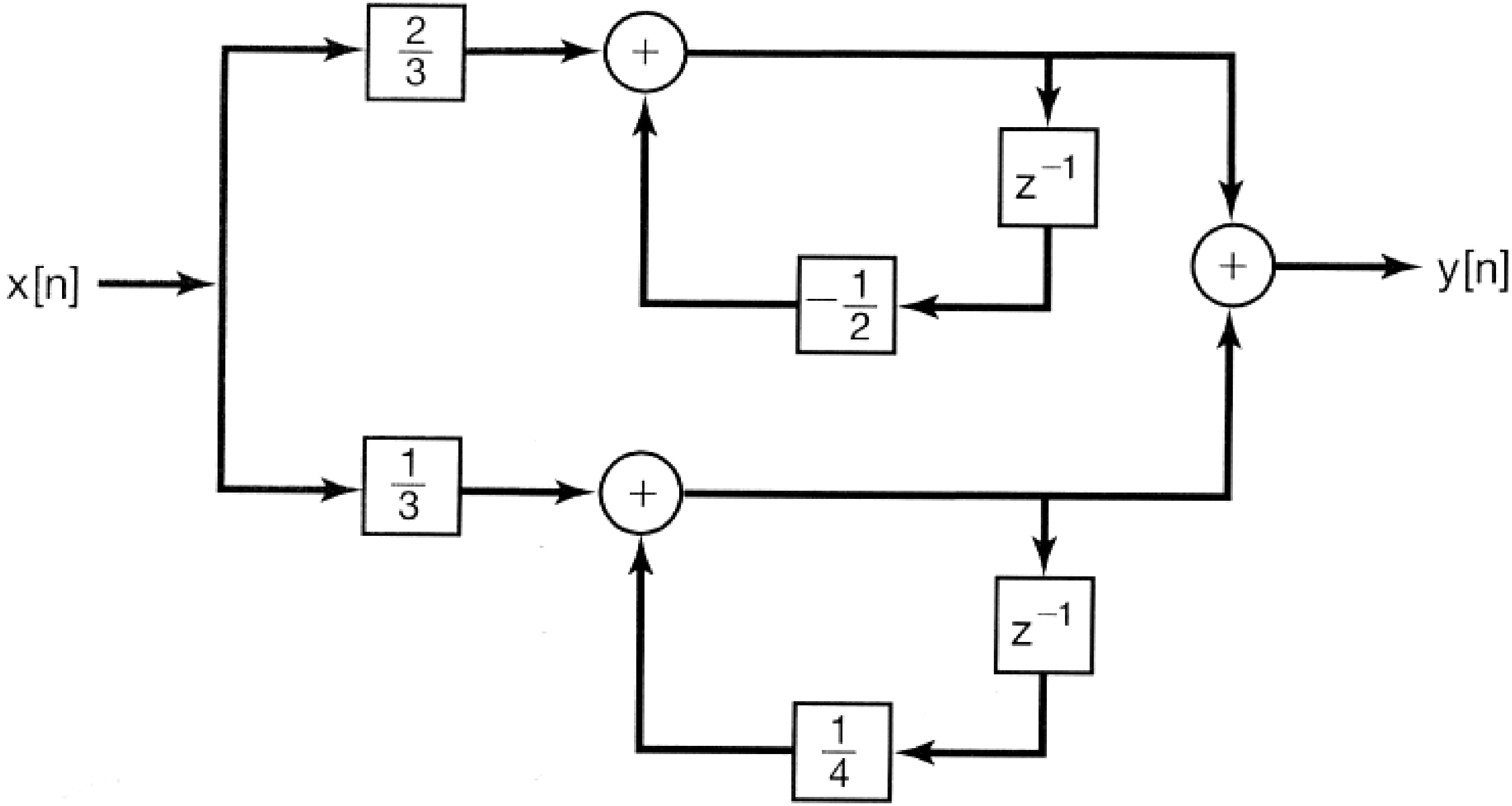


Ex.10.30 Consider the second-order system function

$$H(z)=\frac{1}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}=\frac{1}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}=\frac{2/3}{1+\frac{1}{2}z^{-1}}+\frac{1/3}{1-\frac{1}{4}z^{-1}}$$

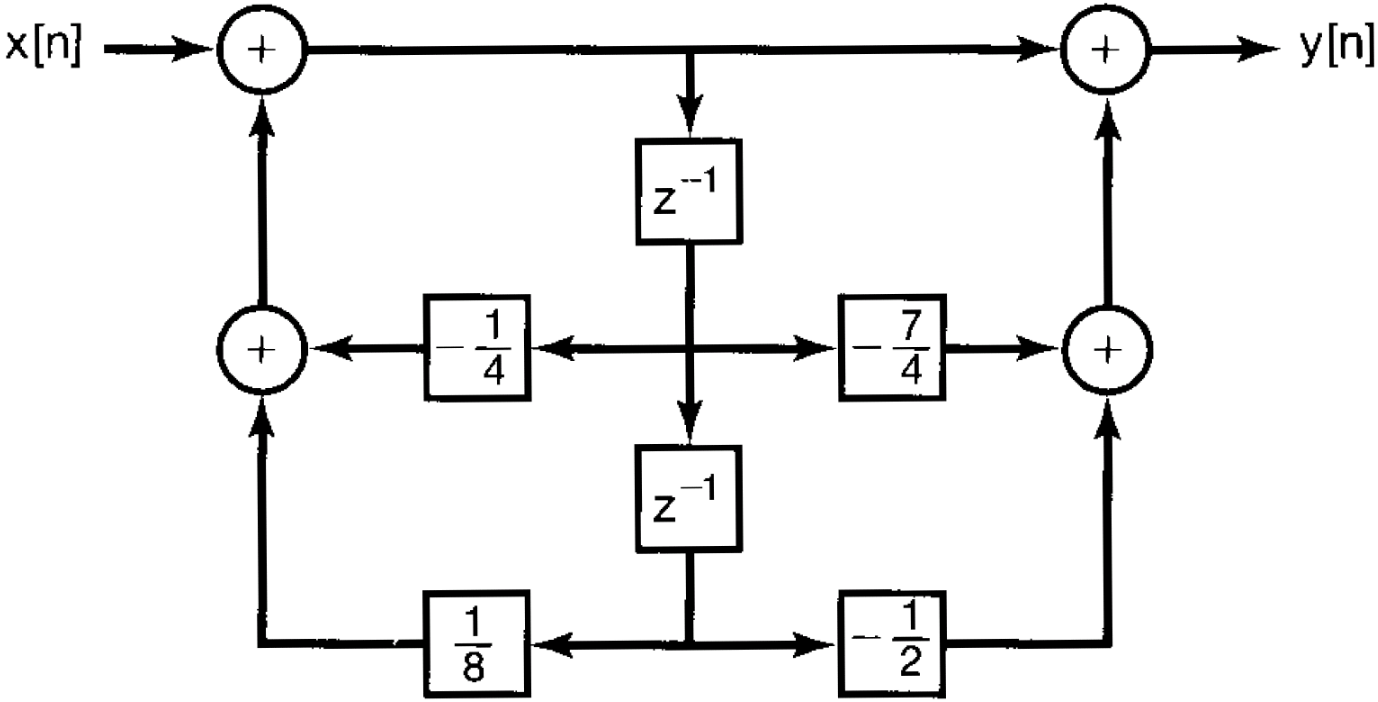



x[n]
+
+
y[n]
z^{-1}
z^{-1}
$-\frac{1}{2}$
$\frac{1}{4}$

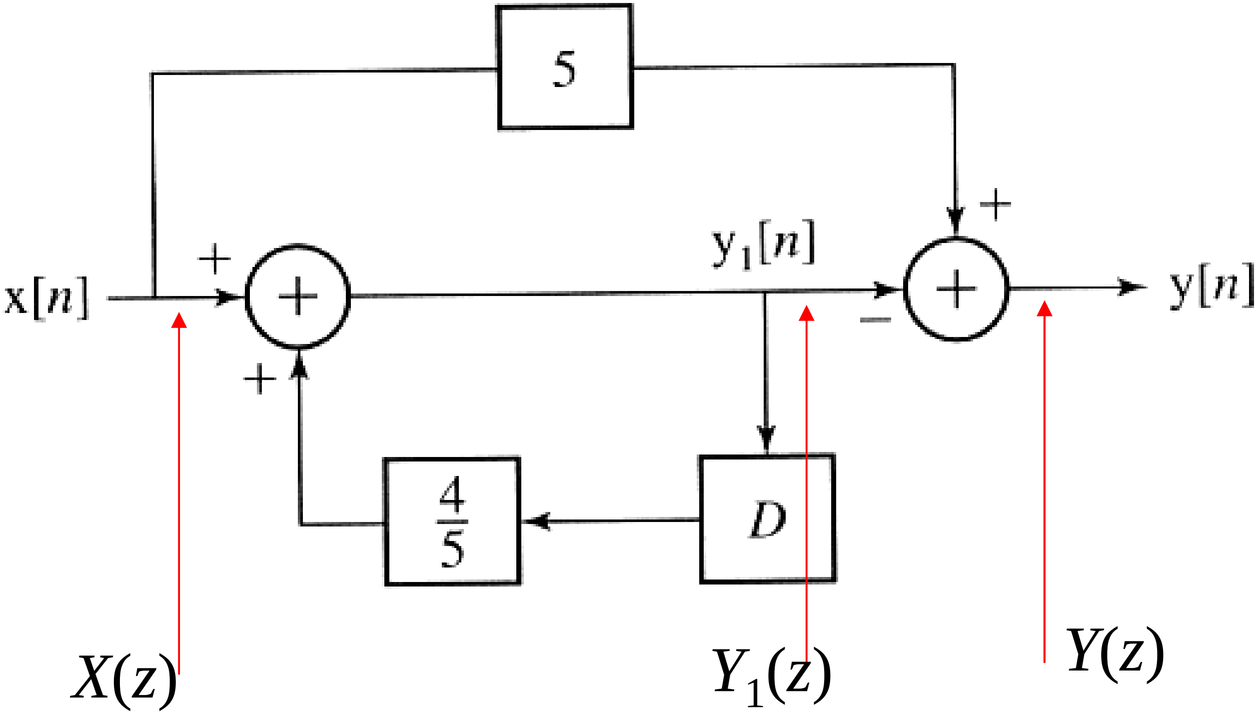

x[n]
$\frac{2}{3}$
+
z^{-1}
$-\frac{1}{2}$
+
y[n]
$\frac{1}{3}$
+
z^{-1}
$\frac{1}{4}$

- Ex. 10.31 $H(z)=\dfrac{1-\frac{7}{4}z^{-1}-\frac{1}{2}z^{-2}}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}=\left(\dfrac{1+\frac{1}{4}z^{-1}}{1+\frac{1}{2}z^{-1}}\right)\left(\dfrac{1-2z^{-1}}{1-\frac{1}{4}z^{-1}}\right)$

$$=4+\frac{5/3}{1+\frac{1}{2}z^{-1}}-\frac{14/3}{1-\frac{1}{4}z^{-1}}$$



➢Ex. A system is shown as follow. Find the transform function.



☞ $Y(z)=5X(z)-Y_1(z)$ and $Y_1(z)=X(z)+\frac{4}{5}z^{-1}Y_1(z)$

- ■ $Y_1(z)=X(z)/(1-\frac{4}{5}z^{-1})$
- ■ $Y(z)=X(z)[5-\frac{1}{(1-\frac{4}{5}z^{-1})}]=X(z)\frac{4-4z^{-1}}{(1-\frac{4}{5}z^{-1})}$

☞ $H(z)=\frac{Y(z)}{X(z)}=\frac{4-4z^{-1}}{(1-\frac{4}{5}z^{-1})}$

■ $y[n]=4\{x[n]-x[n-1]\}+\frac{4}{5}y[n-1]$

- 10.9 The unilateral *z*-transform
  - ➢Bilateral *z*-transform
  - ➢Unilateral *z*-transform
    - ☞That is particularly useful in analyzing causal systems specified by linear constant-coefficient difference equations with nonzero initial conditions.

- ☞Define unilateral *z*-transform of *x*[*n*]

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} x[n]z^{-n}.$$

$$x[n] \overset{\mathcal{UZ}}{\longleftrightarrow} \mathcal{X}(z) = \mathcal{UZ}\{x[n]\}.$$

- ☞ The unilateral *z*-transform of *x*[*n*] can be thought of as the bilateral transform of *x*[*n*]*u*[*n*].
- ☞For any sequence that is zero for *n*<0, the unilateral and bilateral *z*-transforms will be identical.
- ☞Since *x*[*n*]*u*[*n*] is always a right-side sequence, the region of convergence of $\mathcal{X}(z)$ is always the exterior of a circle.

- 10.9.1 (p.790)

Ex. 10.32 Consider the signal $x[n]=a^n u[n]$

- ☞The unilateral and bilateral transforms are equal for this example.

$$\mathcal{X}(z) = \frac{1}{1-az^{-1}}, \qquad |z| > |a|.$$

Ex. 10.33 Let $x[n] = a^{n+1}u[n+1].$

- ☞ The bilateral transform

$$X(z) = \frac{z}{1-az^{-1}}, \qquad |z| > |a|.$$

- ☞ The unilateral transform

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} a^{n+1} z^{-n} = \frac{a}{1-az^{-1}}, \quad |z|>|a|$$

- ☞In this case the unilateral and bilateral are not equal.

- ➢We consider a rational function of *z* written as a ratio of polynomials in *z* (not in z^{-1}) --- $p(z)/q(z)$.
  - ☞For this to be a unilateral transform (with the appropriately chosen ROC as the exterior of a circle), the degree of the numerator must be no bigger than than the degree of the denominator.
    - ■ ①$1/(1-az^{-1})$? ②$z^2/(z-a)$?

- ➢ Ex. $y[n]=x[n-1]$

$$\mathcal{Y}(z) = \sum_{n=0}^{\infty} x[n-1]z^{-n} = x[-1] + \sum_{n=1}^{\infty} x[n-1]z^{-n}$$

$$= x[-1] + \sum_{n=0}^{\infty} x[n]z^{-(n+1)} = x[-1] + z^{-1}\sum_{n=0}^{\infty} x[n]z^{-n},$$

☞ $\mathcal{Y}(z) = x[-1] + z^{-1}\mathcal{X}(z).$

☞ The unilateral transform of $w[n] = y[n-1] = x[n-2]$

- ■ $\mathcal{W}(z) = x[-2] + x[-1]z^{-1} + z^{-2}\mathcal{X}(z).$

☞ $x[n+1] \overset{\mathcal{UZ}}{\longleftrightarrow} z\mathcal{X}(z) - zx[0].$

➢10.9.3 (p.795)

Ex. 10.36, 37 Consider the causal LTI system described by the difference equation $y[n]+3y[n-1]=x[n]$

☞The system function for this system is

$$\mathcal{H}(z) = \frac{1}{1+3z^{-1}}.$$

☞Let the input $x[n]=\alpha u[n]$. The unilateral z-transform of the output $y[n]$ is

$$\mathcal{Y}(z) = \mathcal{H}(z)\mathcal{X}(z) = \frac{\alpha}{(1+3z^{-1})(1-z^{-1})}$$

$$= \frac{(3/4)\alpha}{1+3z^{-1}} + \frac{(1/4)\alpha}{1-z^{-1}}.$$

☞With the initial condition $y[-1]=\beta$.

$$\mathcal{Y}(z) + 3\beta + 3z^{-1}\mathcal{Y}(z) = \frac{\alpha}{1-z^{-1}}.$$

■ $$\mathcal{Y}(z) = -\frac{3\beta}{1+3z^{-1}} + \frac{\alpha}{(1+3z^{-1})(1-z^{-1})}.$$

☞If α=8 and β=1

$$\mathcal{Y}(z) = \frac{3}{1+3z^{-1}} + \frac{2}{1-z^{-1}},$$

■ $y[n] = [3(-3)^n + 2]u[n], \quad \text{for } n \geq 0.$