Chapter 8: Relations

§8.1: Relations and Properties

- Let A, B be any two sets.
- A ***binary relation R from A to B***, written (with signature) $R{:}A\leftrightarrow B$, is a subset of $A\times B$.
  - *E.g.*, let **<** : $\mathbf{N}\leftrightarrow\mathbf{N}$:≡ $\{(n,m) \mid n < m\}$
- The notation $a\,R\,b$ or aRb means $(a,b)\in R$.
  - *E.g.*, a **<** b means $(a,b)\in$ **<**
- If aRb we may say "a is related to b (by relation R)", or "a relates to b (under relation R)".
- A binary relation R corresponds to a predicate function $P_R{:}A\times B\rightarrow\{\mathbf{T},\mathbf{F}\}$ defined over the 2 sets A,B; *e.g.*, "eats" :≡ $\{(a,b) \mid$ organism a eats food $b\}$

Complementary Relations

- Let $R{:}A\leftrightarrow B$ be any binary relation.
- Then, $\not R{:}A\leftrightarrow B$, the *complement* of R, is the binary relation defined by
  $$\not R :\equiv \{(a,b) \mid (a,b)\notin R\} = (A\times B) - R$$
  Note this is just $\overline{R}$ if the universe of discourse is $U = A\times B$; thus the name *complement*.
- Note the complement of $\not R$ is R.

Example: $\not< \; = \{(a,b) \mid (a,b)\notin <\} = \{(a,b) \mid \neg a<b\} = \;\geq$

Relations Example

- Let $A=\{0,1,2\}$, $B = \{a,b\}$. Then $R = \{(0,a), (0,b), (1,a), (2,b)\}$ is a relation $R: A\leftrightarrow B$.

  $0\ R\ a$, $1\ R\ a$, but $1\ \not R\ b$.

R	a	b
0		
1		
2		

Inverse Relations

- Any binary relation $R:A\leftrightarrow B$ has an *inverse relation* $R^{-1}:B\leftrightarrow A$, defined by

  $$R^{-1} :\equiv \{(b,a) \mid (a,b)\in R\}.$$

  E.g., $<^{-1} = \{(b,a) \mid a<b\} = \{(b,a) \mid b>a\} = >$.
- *E.g.*, if R:People→Foods is defined by
  $aRb \Leftrightarrow a \text{ eats } b$, then:
  $b\,R^{-1}\,a \Leftrightarrow b \text{ is eaten by } a$. (Passive voice.)

Relations on a Set

- A (binary) relation from a set A to itself is called a relation *on* the set A.
- *E.g.*, the "**<**" relation from earlier was defined as a relation *on* the set **N** of natural numbers.
- The *identity relation* $\mathbf{I}_A$ on a set A is the set $\{(a,a)|a\in A\}$.

Example

- $A = \{1,2,3,4\}$ $R = \{(a,b) \mid a$ divides $b\} = ?$.

  $\boldsymbol{R} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

R	1	2	3	4
1				
2				
3				
4				

Example

- *Q*: How many relations are there on a set with n elements?

  A: 1. What is the cardinality of $A \times A$ if A has n elements?

  2. How many subsets are there for $A \times A$?

Reflexivity

- A relation R on A is *reflexive* if $\forall a \in A, aRa$.
  - *E.g.*, the relation $\geq :\equiv \{(a,b) \mid a \geq b\}$ is reflexive.
- A relation is *irreflexive* iff its complementary relation is reflexive.
  - Note "*irreflexive*" $\neq$ "*not reflexive*"!
  - Example: < is irreflexive.
  - Note: "likes" between people is not reflexive, but not irreflexive either. (Not everyone likes themselves, but not everyone dislikes themselves either.)

Symmetry & Antisymmetry

- A binary relation R on A is symmetric iff $R = R^{-1}$, that is, if $(a,b)\in R \leftrightarrow (b,a)\in R$.
  - *E.g.*, = (equality) is symmetric. < is not.
  - "is married to" is symmetric, "likes" is not.
- A binary relation R is *antisymmetric* if $(a,b)\in R \rightarrow (b,a)\notin R$.
  - < is antisymmetric, "likes" is not.

Transitivity

- A relation R is *transitive* iff (for all a,b,c)
  $(a,b)\in R \land (b,c)\in R \rightarrow (a,c)\in R$.
- A relation is *intransitive* if it is not transitive.
- Examples: "is an ancestor of" is transitive.
- "likes" is intransitive.
- "is within 1 mile of" is… ?

Example

- $R_1 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$
- $R_2 = \{(1,1),(1,2),(2,1)\}$
- $R_3 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),$
  $(3,3),(3,4),(4,4)\}$
- Which of the relations are transitive?

Totality

- A relation R:$A \leftrightarrow B$ is *total* if for every $a \in A$, there is at least one $b \in B$ such that $(a,b) \in R$.
- If R is not total, then it is called *strictly partial*.
- A *partial relation* is a relation that *might* be strictly partial. Or, it might be total. (In other words, all relations are considered "partial.")

Example

- A = {1,2,3,4} R= {(a,b)| a divides b}=?.

  R = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}. Is R totality?

R	1	2	3	4
1				
2				
3				
4				

Functionality

- A relation $R{:}A\leftrightarrow B$ is *functional* (that is, it is also a partial function $R{:}A\rightarrow B$) if, for any $a\in A$, there is *at most 1* $b\in B$ such that $(a,b)\in R$.
- R is *antifunctional* if its inverse relation R^{-1} is functional.
  - Note: A functional relation (partial function) that is also antifunctional is an invertible partial function.
- R is a *total function* $R{:}A\rightarrow B$ if it is both functional and total, that is, for any $a\in A$, there is *exactly* 1 b such that $(a,b)\in R$. If R is functional but not total, then it is a *strictly partial function*.

Composite Relations

- Let $R:A\leftrightarrow B$, and $S:B\leftrightarrow C$. Then the *composite* $S\circ R$ of R and S is defined as:

$$S\circ R = \{(a,c) \mid aRb \wedge bSc\}$$

- Note function composition $f\circ g$ is an example.
- The n^{th} power R^n of a relation R on a set A can be defined recursively by:

$$R^0 :\equiv \mathbf{I}_A ; \quad R^{n+1} :\equiv R^n\circ R \quad \text{for all } n\geq 0.$$

  – Negative powers of R can also be defined if desired, by $R^{-n} :\equiv (R^{-1})^n$.

Example

- Find the composite $S \circ R$, where

$$R = \{(1,1),(1,4),(2,3),(3,1),(3,4)\}$$

$$S = \{(1,0),(2,0),(3,1),(3,2),(4,1)\}$$

$$S \circ R = \{ \qquad \}$$

Example

- Find the powers R^n, where

$$R = \{(1,1),(2,1),(3,2),(4,3)\}.$$

$$R^2 = \{ \qquad \}$$

$$R^3 = \{ \qquad \}$$

$$R^4 = \{ \qquad \}$$

$$= R^4 = R^5 = \cdots$$

Theorem:
The relation R on a set A is *transitive* iff $R^n \subseteq R$ for n=1,2,…

§8.2: n-ary Relations

- An n-ary relation R on sets $A_1,\ldots,A_n$, written $R:A_1,\ldots,A_n$, is a subset $R \subseteq A_1 \times \ldots \times A_n$.
- The sets A_i are called the *domains* of R.
- The *degree* of R is n.
- R is *functional in domain* A_i if it contains at most one n-tuple $(\ldots, a_i, \ldots)$ for any value a_i within domain A_i.

Relational Databases

- A *relational database* is essentially an n-ary relation R.
- A domain A_i is a *primary key* for the database if the relation R is functional in A_i.
- A *composite key* for the database is a set of domains $\{A_i, A_j, \ldots\}$ such that R contains at most 1 n-tuple $(\ldots,a_i,\ldots,a_j,\ldots)$ for each composite value $(a_i, a_j,\ldots)\in A_i \times A_j \times \ldots$

Example

- $A_1 = \{a,b,c\}$, $A_2 = \{x,y,z\}$, $A_3 = \{1,2,3,4\}$, which is a primary key?

A_1	A_2	A_3
a	x	1
a	y	2
b	z	3
c	z	4

Selection Operators

- Let A be any n-ary domain $A=A_1\times\ldots\times A_n$, and let $C:A\rightarrow\{\mathbf{T},\mathbf{F}\}$ be any condition (predicate) on elements (n-tuples) of A.
- Then, the *selection operator* s_C is the operator that maps any (n-ary) relation R on A to the n-ary relation of all n-tuples from R that satisfy C.
  - *I.e.*, $\forall R\subseteq A$, $s_C(R) = R\cap\{a\in A \mid s_C(a) = \mathbf{T}\}$

Selection Operator Example

- Suppose we have a domain A = StudentName × Standing × SocSecNos
- Suppose we define a certain condition on A, *UpperLevel*(*name*,*standing*,*ssn*) :≡ [(*standing* = junior) ∨ (*standing* = senior)]
- Then, $s_{UpperLevel}$ is the selection operator that takes any relation R on A (database of students) and produces a relation consisting of *just* the upper-level classes (juniors and seniors).

Projection Operators

- Let $A = A_1\times\ldots\times A_n$ be any n-ary domain, and let $\{i_k\}=(i_1,\ldots,i_m)$ be a sequence of indices all falling in the range 1 to n,
  - That is, where $1 \le i_k \le n$ for all $1 \le k \le m$.
- Then the *projection operator* on n-tuples

$$P_{\{i_k\}} : A \rightarrow A_{i_1} \times \ldots \times A_{i_m}$$

is defined by:

$$P_{\{i_k\}}(a_1,\ldots,a_n) = (a_{i_1},\ldots,a_{i_m})$$

Projection Example

- Suppose we have a ternary (3-ary) domain $Cars = Model \times Year \times Color$. (note n=3).
- Consider the index sequence $\{i_k\}$= 1,3. (m=2)
- Then the projection $P_{\{i_k\}}$ simply maps each tuple $(a_1,a_2,a_3) = (model,year,color)$ to its image:

$$(a_{i_1}, a_{i_2}) = (a_1, a_3) = (model, color)$$

- This operator can be usefully applied to a whole relation $R \subseteq Cars$ (database of cars) to obtain a list of model/color combinations available.

Projection Example

- What results when the projection $P_{1,3}$ is applied to the 4-tuples (2,3,0,4), and (a_1,a_2,a_3,a_4)?

Join Operator

- Puts two relations together to form a sort of combined relation.
- If the tuple (A,B) appears in R_1, and the tuple (B,C) appears in R_2, then the tuple (A,B,C) appears in the join $J(R_1,R_2)$.
  - A, B, C can also be sequences of elements rather than single elements.

Join Operator

- If the tuple (A,B) appears in R_1, and the tuple (B,C) appears in R_2, then the tuple (A,B,C) appears in the join $J(R_1,R_2)$.
- R:degree of m; S:degree of n;

  $J_p(R,S)=(A,B,C)$: degree of $m+n-p$,

  where $R=(A,B)$, $S=(B,C)$, $|B|=p$

Join Example

- Suppose R_1 is a teaching assignment table, relating *Professors* to *Courses*.
- Suppose R_2 is a room assignment table relating *Courses* to *Rooms*,*Times*.
- Then $J(R_1,R_2)$ is like your class schedule, listing (*professor*,*course*,*room*,*time*).

§8.3: Representing Relations

- Some ways to represent *n*-ary relations:
  - With an explicit list or table of its tuples.
  - With a function from the domain to **{T,F}**.
    - Or with an algorithm for computing this function.
- Some special ways to represent binary relations:
  - With a zero-one matrix.
  - With a directed graph.

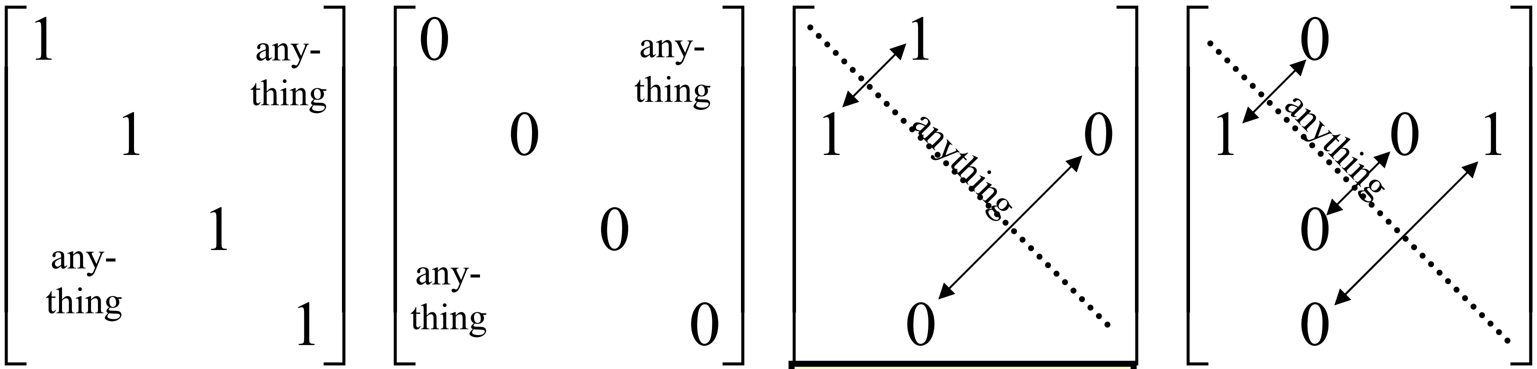
Using Zero-One Matrices

- To represent a relation R by a matrix $\mathbf{M}_R = [m_{ij}]$, let $m_{ij} = 1$ if $(a_i,b_j)\in R$, else 0.
- *E.g.*, Joe likes Susan and Mary, Fred likes Mary, and Mark likes Sally.
- The 0-1 matrix representation of that "Likes" relation:

	Susan	Mary	Sally
Joe	1	1	0
Fred	0	1	0
Mark	0	0	1

Zero-One Reflexive, Symmetric

- Terms: *Reflexive*, *non-Reflexive*, *irreflexive*, *symmetric*, *asymmetric*, *and antisymmetric*.
  - These relation characteristics are very easy to recognize by inspection of the zero-one matrix.



Reflexive: all 1's on diagonal	*Irreflexive*: all 0's on diagonal	*Symmetric*: all identical across diagonal	*Antisymmetric*: all 1's are across from 0's

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Examples

Ex: Is R reflexive, symmetric, and/or antisymmetric?

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Ex: What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$?

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$$

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$$

$$M_{R_1 \circ R_2} = M_{R_2} \odot M_{R_1}$$

Examples

Ex: What are the matrices representing

R^2 ?

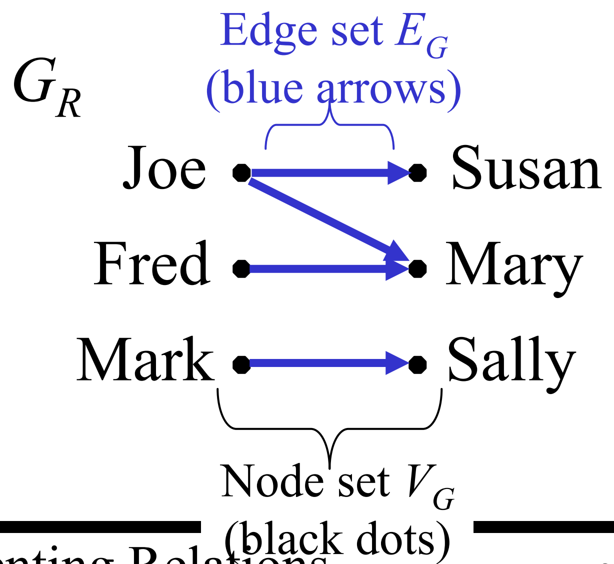
$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$M_{R^2} = M_R \odot M_R = M_R^{[2]} =$$

Using Directed Graphs

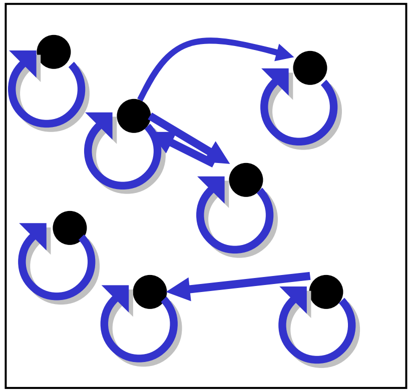
- A *directed graph* or *digraph* $G=(V_G,E_G)$ is a set V_G of *vertices (nodes)* with a set $E_G \subseteq V_G \times V_G$ of *edges (arcs,links)*. Visually represented using dots for nodes, and arrows for edges. Notice that a relation $R:A \leftrightarrow B$ can be represented as a graph $G_R=(V_G=A \cup B,\ E_G=R)$.

$\mathbf{M}_R$	Susan	Mary	Sally
Joe	1	1	0
Fred	0	1	0
Mark	0	0	1

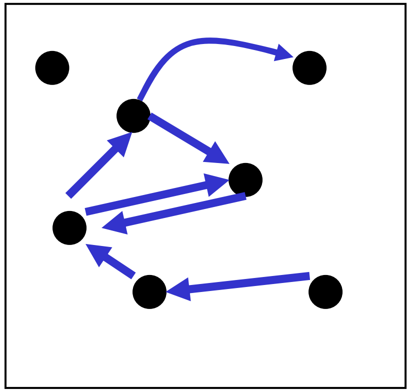


Digraph Reflexive, Symmetric

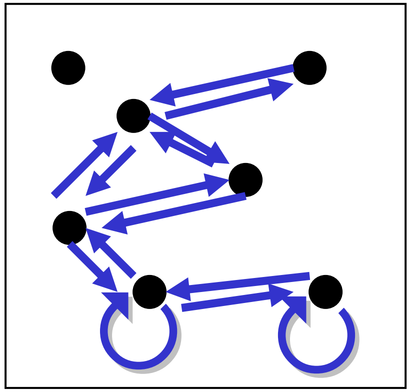
It is extremely easy to recognize the reflexive/irreflexive/ symmetric/antisymmetric properties by graph inspection.



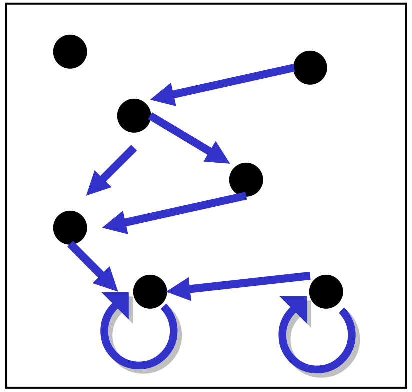
Reflexive: Every node has a self-loop



Irreflexive: No node links to itself



Symmetric: Every link is bidirectional



Antisymmetric: No link is bidirectional

Asymmetric, non-antisymmetric

Non-reflexive, non-irreflexive

§8.4: Closures of Relations

- For any property X, the "X closure" of a set A is defined as the "smallest" superset of A that has the given property.
- The *reflexive closure* of a relation R on A is obtained by adding (a,a) to R for each $a\in A$. *I.e.*, it is $R \cup I_A$
- The *symmetric closure* of R is obtained by adding (b,a) to R for each (a,b) in R. *I.e.*, it is $R \cup R^{-1}$
- The *transitive closure* or *connectivity relation* of R is obtained by repeatedly adding (a,c) to R for each (a,b),(b,c) in R.
  - *I.e.*, it is $$R^* = \bigcup_{n\in\mathbf{Z}^+} R^n$$

Examples

- What is the reflexive closure of the relation

  $R = \{(a,b) \mid a < b\}$ on the set of integers ?

  $R \cup I_A = \{(a,b) \mid a < b\} \cup \{(a,a) \mid a \in Z\} = \{(a,b) \mid a \leq b\}$
- What is the *symmetric closure* of $R = \{(a,b) \mid a > b\}$ on the set of positive integer?

  $R \cup R^{-1} = \{(a,b) \mid a > b\} \cup \{(a,b) \mid a < b\} = \{(a,b) \mid a \neq b\}$
- The *transitive closure* or *connectivity relation* of R

  $R = \{(1,3),(1,4),(2,1),(3,2)\}$, $R^2 = \{ \quad \}$

$R^3 = \{ \quad \}$, $R^4 = \{ \quad \}$

So $R^* = \{ \quad \}$

Paths in Digraphs/Binary Relations

- A *path* of length n from node a to b in the directed graph G (or the binary relation R) is a sequence $(a,x_1), (x_1,x_2), \ldots, (x_{n-1},b)$ of n ordered pairs in E_G (or R).
  - An empty sequence of edges is considered a path of length 0 from a to a.
  - If any path from a to b exists, then we say that *a is connected to b*. ("You can get there from here.")
- A path of length $n \geq 1$ from a to a is called a *circuit* or a cycle.
- Note that there exists a path of length n from a to b in R if and only if $(a,b) \in R^n$.

Simple Transitive Closure Alg.

A procedure to compute R^* with 0-1 matrices.

procedure *transClosure*($\mathbf{M}_R$:rank-n 0-1 mat.)

$\mathbf{A} := \mathbf{B} := \mathbf{M}_R$;

for $i := 2$ to n **begin**

$\mathbf{A} := \mathbf{A} \odot \mathbf{M}_R$; $\mathbf{B} := \mathbf{B} \vee \mathbf{A}$ {join}

end   {note **A** represents R^i}

return B   {Alg. takes $\Theta(n^4)$ time}

A Faster Transitive Closure Alg.

procedure *transClosure*($\mathbf{M}_R$:rank-n 0-1 mat.)

  $\mathbf{A} := \mathbf{B} := \mathbf{M}_R$;

  for $i := 1$ to $\lceil \log_2 n \rceil$ **begin**

    $\mathbf{A} := \mathbf{A} \odot \mathbf{A}$;   {**A** represents R^{2^i}}

    $\mathbf{B} := \mathbf{B} \vee \mathbf{A}$ {"add" $\mathbf{M}_R^{[2^i]}$ into **B**}

  end

  return B {Alg. takes only $\Theta(n^3 \log n)$ time}

Roy-Warshall Algorithm

- Uses only $\Theta(n^3)$ operations!

Procedure *Warshall*($\mathbf{M}_R$: rank-n 0-1 matrix)

$\mathbf{W} := \mathbf{M}_R$

for $k := 1$ **to** n

for $i := 1$ **to** n

for $j := 1$ **to** n

$w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj})$

return W {this represents R^*}

$w_{ij} = 1$ means there is a path from i to j going only through nodes $\leq k$

§8.5: Equivalence Relations

- An *equivalence relation* (e.r.) on a set A is simply any binary relation on A that is reflexive, symmetric, and transitive.
  - *E.g.*, = itself is an equivalence relation.
  - For any function $f{:}A\rightarrow B$, the relation "have the same f value", or $=_f :\equiv \{(a_1,a_2) \mid f(a_1)=f(a_2)\}$ is an equivalence relation, *e.g.*, let m= "mother of" then $=_m$ = "have the same mother" is an e.r.

Equivalence Relation Examples

- "Strings a and b are the same length."
- "Integers a and b have the same absolute value."
- "Real numbers a and b have the same fractional part (*i.e.*, $a - b \in \mathbf{Z}$)."
- "Integers a and b have the same residue modulo m." (for a given m>1)

Equivalence Classes

- Let R be any equiv. rel. on a set A.
- The *equivalence class* of a,

  $[a]_R :\equiv \{ b \mid aRb \}$ (optional subscript R)

  - It is the set of all elements of A that are "equivalent" to a according to the eq.rel. R.
  - Each such b (including a itself) is called a *representative* of $[a]_R$.
- Since $f(a)=[a]_R$ is a function of a, any equivalence relation R be defined using $aRb :\equiv$ "a and b have the same f value", given that f.

Equivalence Class Examples

- "Strings a and b are the same length."
  - $[a]$ = the set of all strings of the same length as a.
- "Integers a and b have the same absolute value."
  - $[a]$ = the set $\{a, -a\}$
- "Real numbers a and b have the same fractional part (*i.e.*, $a - b \in \mathbf{Z}$)."
  - $[a]$ = the set $\{\ldots, a-2, a-1, a, a+1, a+2, \ldots\}$
- "Integers a and b have the same residue modulo m." (for a given $m>1$)
  - $[a]$ = the set $\{\ldots, a-2m, a-m, a, a+m, a+2m, \ldots\}$

Partitions

- A *partition* of a set A is the set of all the equivalence classes $\{A_1, A_2, \ldots\}$ for some e.r. on A.
- The A_i's are all disjoint and their union = A.
- They "partition" the set into pieces. Within each piece, all members of the set are equivalent to each other.