Chapter 4: Induction and Recursion

§4.1: Mathematical Induction

- A powerful, rigorous technique for proving that a predicate $P(n)$ is true for *every positive integer n*, no matter how large.
- Essentially a "domino effect" principle.
- Based on a predicate-logic inference rule:

$$\begin{array}{l} P(1) \\ \forall k\geq 1\ (P(k)\rightarrow P(k+1)) \\ \hline \therefore \forall n\geq 1\ P(n) \end{array}$$

"The First Principle of Mathematical Induction"

Outline of an Inductive Proof

- Want to prove $\forall n\, P(n)$…
- *Base case* (or *basis step*): Prove $P(1)$.
- *Inductive step*: Prove $\forall k\, P(k) \rightarrow P(k+1)$.
  - *E.g.* use a direct proof:
  - Let $k \in \mathbf{N}$, assume $P(k)$. (*inductive hypothesis*)
  - Under this assumption, prove $P(k+1)$.
- Inductive inference rule then gives $\forall n\, P(n)$.

Induction Example (1st princ.)

- Prove that the sum of the first n odd positive integers is n^2. That is, prove:

$$\underbrace{\forall n \geq 1 : \sum_{i=1}^{n} (2i-1) = n^2}_{P(n)}$$

- Proof by induction.
  - Base case: Let n=1. The sum of the first 1 odd positive integer is 1 which equals 1^2. (Cont…)

Example cont.

- Inductive step: Prove $\forall k \geq 1$: $P(k) \rightarrow P(k+1)$.
  - Let $k \geq 1$, assume $P(k)$, and prove $P(k+1)$.

$$\sum_{i=1}^{k+1}(2i-1) = \left(\sum_{i=1}^{k}(2i-1)\right) + (2(k+1)-1)$$

$$= k^2 + 2k + 1$$

By inductive hypothesis P(k)

$$= (k+1)^2$$

Another Induction Example

- Prove that $\forall n>0, n<2^n$. Let $P(n)=(n<2^n)$
  - Base case: $P(1)=(1<2^1)=(1<2)=\mathbf{T}$.
  - Inductive step: For $k>0$, prove $P(k)\rightarrow P(k+1)$.
    - Assuming $k<2^k$, prove $k+1 < 2^{k+1}$.
    - Note $k + 1 < 2^k + 1$ (by inductive hypothesis)
      $< 2^k + 2^k$ (because $1<2=2\cdot 2^0\leq 2\cdot 2^{k-1}= 2^k$)
      $= 2^{k+1}$
    - So $k + 1 < 2^{k+1}$, and we're done.

Validity of Induction

Proof that $\forall k \geq 1\ P(k)$ is a valid consequent:

Given any $k \geq 1$, $\forall n \geq 1\ (P(n) \rightarrow P(n+1))$ (antecedent 2) trivially implies $\forall n \geq 1\ (n<k) \rightarrow (P(n) \rightarrow P(n+1))$, or $(P(1) \rightarrow P(2)) \wedge (P(2) \rightarrow P(3)) \wedge \ldots \wedge (P(k-1) \rightarrow P(k))$. Repeatedly applying the hypothetical syllogism rule to adjacent implications k-1 times then gives $P(1) \rightarrow P(k)$; which with $P(1)$ (antecedent #1) and *modus ponens* gives $P(k)$. Thus $\forall k \geq 1\ P(k)$.

The Well-Ordering Property

- The validity of the inductive inference rule can also be proved using the *well-ordering property*, which says:
  - Every non-empty set of non-negative integers has a minimum (smallest) element.
  - $\forall\ \emptyset \subset S \subseteq \mathbf{N} : \exists m \in S : \forall n \in S : m \leq n$
- Implies $\{n | \neg P(n)\}$ has a min. element m, but then $P(m\text{-}1) \rightarrow P((m\text{-}1)+1)$ contradicted.

Generalizing Induction

- Can also be used to prove $\forall n \geq c\ P(n)$ for a given constant $c \in \mathbf{Z}$, where maybe $c \neq 1$.
  - In this circumstance, the base case is to prove $P(c)$ rather than $P(1)$, and the inductive step is to prove $\forall k \geq c\ (P(k) \rightarrow P(k+1))$.
- Induction can also be used to prove $\forall n \geq c\ P(a_n)$ for an arbitrary series $\{a_n\}$.
- Can reduce these to the form already shown.

§4.2： Strong Induction

- Characterized by another inference rule:

$$\begin{array}{l} P(1) \\ \forall k\geq 1\text{: } \big(\overbrace{\forall 1\leq i\leq k\ P(i)}^{P \text{ is true in } all \text{ previous cases}}\big) \rightarrow P(k+1) \\ \hline \therefore \forall n\geq 1\text{: } P(n) \end{array}$$

- Difference with 1st principle is that the inductive step uses the fact that $P(i)$ is true for *all* smaller $i<k+1$, not just for $i=k$.

Example of Second Principle

- Show that every $n>1$ can be written as a product $p_1p_2\ldots p_s$ of some series of s prime numbers. Let $P(n)$="n has that property"
- Base case:
- Inductive step: Let $k\geq 2$. Assume $\forall 2\leq i\leq k$: $P(i)$. Consider $k+1$. If prime,

Else $k+1=ab$, where $1<a\leq k$ and $1<b\leq k$.

Another 2nd Principle Example

- Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.
- Base case:
- Inductive step: Let $k \geq 15$, assume $\forall 12 \leq i \leq k$ $P(i)$.

Proofs By Well-Ordering Property

- Use the well-ordering property to prove the division algorithm: $a = dq + r$, $0 \le r < |d|$, where q and r are unique.
  - $S = \{ n \mid n = a - dq \}$ is nonempty, so S has a least element $r = a - dq_0$. If $r \ge 0$, it is also the case that $r < d$. If it were not,
  - If $a = dq_1 + r_1 = dq_2 + r_2$, $0 \le r_1 , r_2 < |d|$, , then

§ 4.3：Recursive Definitions

- In induction, we *prove* all members of an infinite set have some property *P* by proving the truth for larger members in terms of that of smaller members.
- In *recursive definitions*, we similarly *define* a function, a predicate or a set over an infinite number of elements by defining the function or predicate value or set-membership of larger elements in terms of that of smaller ones.

Recursion

- *Recursion* is a general term for the practice of defining an object in terms of *itself* (or of part of itself).
- An inductive proof establishes the truth of $P(n+1)$ *recursively* in terms of $P(n)$.
- There are also recursive *algorithms*, *definitions*, *functions*, *sequences*, and *sets*.

Recursively Defined Functions

- Simplest case: One way to define a function f:**N**$\rightarrow S$ (for any set S) or series $a_n = f(n)$ is to:
  - Define $f(0)$.
  - For $n>0$, define $f(n)$ in terms of $f(0),\ldots,f(n-1)$.
- *E.g.*: Define the series $a_n :\equiv 2^n$ recursively:
  - Let $a_0 :\equiv 1$.
  - For $n>0$, let $a_n :\equiv 2a_{n-1}$.

Another Example

- Suppose we define $f(n)$ for all $n \in \mathbf{N}$ recursively by:
  - Let $f(0)=3$
  - For all $n \in \mathbf{N}$, let $f(n+1)=2f(n)+3$
- What are the values of the following?
  - $f(1)=$ ___ , $f(2)=$ ___ , $f(3)=$ ___ , $f(4)=$

Recursive definition of Factorial

- Give an inductive definition of the factorial function $F(n) :\equiv n! :\equiv 2 \cdot 3 \cdot \ldots \cdot n$.
  - Base case: $F(0) :\equiv 1$
  - Recursive part: $F(n) :\equiv n \cdot F(n\text{-}1)$.
    - $F(1)=1$
    - $F(2)=2$
    - $F(3)=6$

The Fibonacci Series

- The *Fibonacci series* $f_{n\geq 0}$ is a famous series defined by:

$$f_0 :\equiv 0, \quad f_1 :\equiv 1, \quad f_{n\geq 2} :\equiv f_{n-1} + f_{n-2}$$

$$\begin{tikzcd}
0 \arrow[dr] & & & \\
1 \arrow[r] \arrow[dr] & 1 \arrow[d] \arrow[dr] & & \\
 & 2 \arrow[r] \arrow[dr] & 3 \arrow[d] \arrow[dr] & \\
 & & 5 \arrow[r] \arrow[dr] & 8 \arrow[d] \\
 & & & 13
\end{tikzcd}$$

Leonardo Fibonacci
1170-1250

Inductive Proof about Fib. series

- **Theorem:** $f_n < 2^n$. ← Implicitly for all $n \in \mathbf{N}$
- **Proof:** By induction.

Base cases:

Inductive step: Use 2nd principle of induction (strong induction). Assume $\forall i<k,\ f_i < 2^i$. Then

Recursively Defined Sets

- An infinite set S may be defined recursively, by giving:
  - A small finite set of *base* elements of S.
  - A rule for constructing new elements of S from previously-established elements.
  - Implicitly, S has no other elements but these.
- **Example:** Let $3 \in S$, and let $x+y \in S$ if $x,y \in S$. What is S ?

The Set of All Strings

- Given an alphabet Σ, the set Σ^* of all strings over Σ can be recursively defined as:

  $\varepsilon \in \Sigma^*$ ($\varepsilon :\equiv$ "", the empty string) Book uses λ

  $w \in \Sigma^* \wedge x \in \Sigma \rightarrow wx \in \Sigma^*$

- Exercise: Prove that this definition is equivalent to our old one: $\Sigma^* :\equiv \bigcup_{n \in \mathbf{N}} \Sigma^n$

§4.4：Recursive Algorithms

- Recursive definitions can be used to describe *algorithms* as well as functions and sets.
- Example: A procedure to compute a^n.

procedure *power*($a \neq 0$: real, $n \in \mathbf{N}$)

  if $n = 0$ **then return** 1

  else return $a \cdot power(a, n-1)$

Efficiency of Recursive Algorithms

- The time complexity of a recursive algorithm may depend critically on the number of recursive calls it makes.
- Example: *Modular exponentiation* to a power n can take log(n) time if done right, but linear time if done slightly differently.
  - Task: Compute b^n **mod** m, where $m \geq 2$, $n \geq 0$, and $1 \leq b < m$.

Modular Exponentiation Alg. #1

Uses the fact that $b^n = b \cdot b^{n-1}$ and that $x \cdot y \textbf{ mod } m = x \cdot (y \textbf{ mod } m) \textbf{ mod } m$. (Prove the latter theorem at home.)

procedure $mpower(b \geq 1, n \geq 0, m > b \in \mathbf{N})$

{Returns b^n **mod** m.}

if $n=0$ **then return** 1 **else**

return $(b \cdot mpower(b, n-1, m))$ **mod** m

Note this algorithm takes $\Theta(n)$ steps!

Modular Exponentiation Alg. #2

- Uses the fact that $b^{2k} = b^{k\cdot 2} = (b^k)^2$.

procedure *mpower*(b,n,m) {same signature}

if n=0 **then return** 1

else if $2|n$ **then**

return $mpower(b,n/2,m)^2$ **mod** m

else return ($mpower(b,n-1,m)\cdot b$) **mod** m

What is its time complexity? $\Theta(\log n)$ steps

A Slight Variation

Nearly identical but takes $\Theta(n)$ time instead!

procedure *mpower*(b,n,m) {same signature}

  if n=0 **then return** 1

  else if $2|n$ **then**

    return ($mpower(b,n/2,m)\cdot$
     $mpower(b,n/2,m)$) **mod** m

  else return ($mpower(b,n-1,m)\cdot b$) **mod** m

The number of recursive calls made is critical.

Recursive Euclid's Algorithm

procedure *gcd*(a,b∈**N**)
  if $a = 0$ **then return** b
  else return *gcd*(b **mod** a, a)

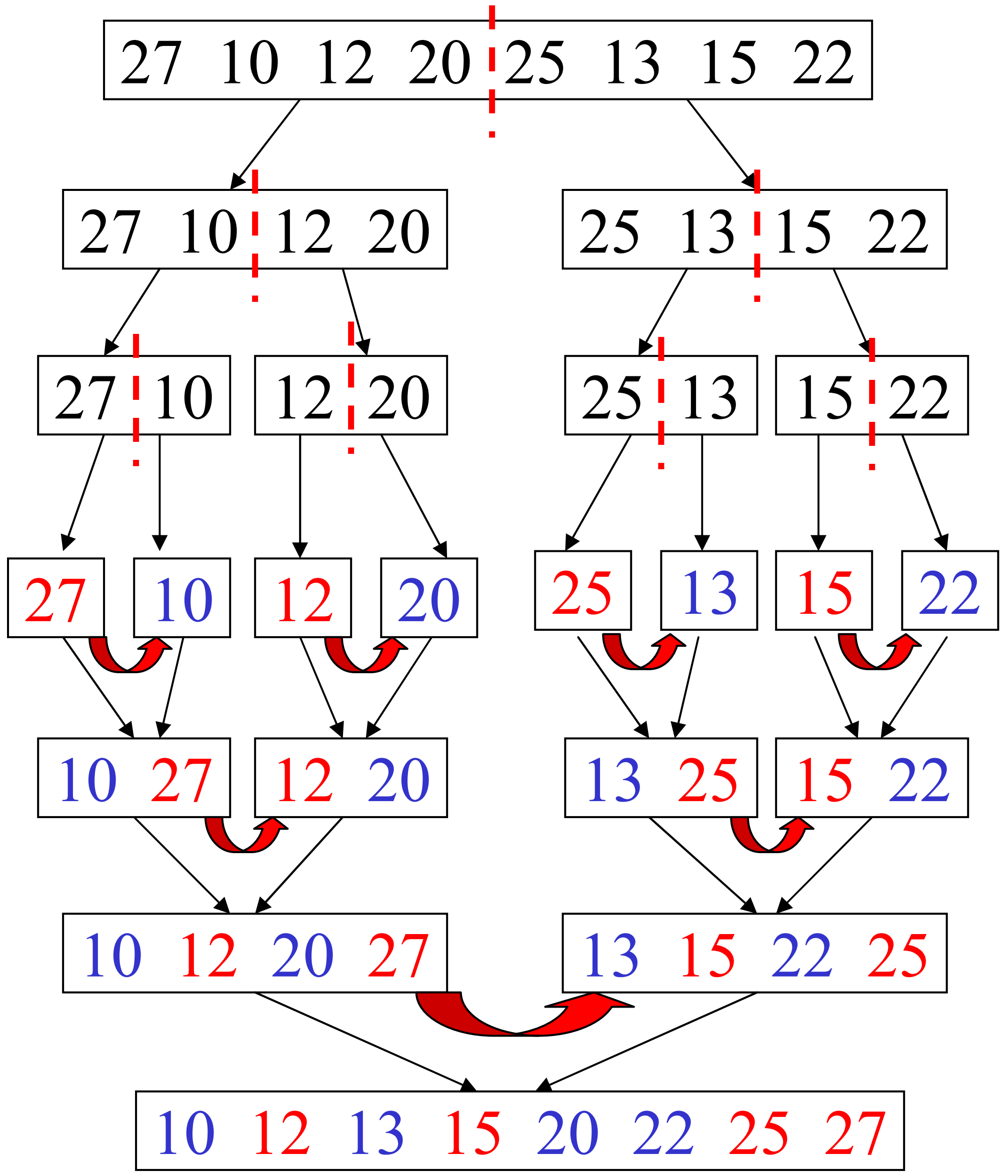
- Note recursive algorithms are often simpler to code than iterative ones…
- However, they can consume more stack space, if your compiler is not smart enough.

Merge Sort

procedure $sort(L = \ell_1,\ldots, \ell_n)$
  if $n>1$ **then**
    $m := \lfloor n/2 \rfloor$ {this is rough ½-way point}
    $L := merge(sort(\ell_1,\ldots, \ell_m),$
        $sort(\ell_{m+1},\ldots, \ell_n))$
  return L

- The merge takes $\Theta(n)$ steps, and merge-sort takes $\Theta(n \log n)$.

Example: Sort the list 27, 10, 12, 20, 25, 13, 15, 22.



Merge Routine

procedure *merge*(A, B: sorted lists)
  L = empty list
  i:=0, j:=0, k:=0
  while $i<|A| \wedge j<|B|$ {$|A|$ is length of A}
    if $i=|A|$ **then** $L_k := B_j$; $j := j + 1$
    else if $j=|B|$ **then** $L_k := A_i$; $i := i + 1$
    else if $A_i < B_j$ **then** $L_k := A_i$; $i := i + 1$
    else $L_k := B_j$; $j := j + 1$
    $k := k+1$
  return L

Takes $\Theta(|A|+|B|)$ time