Chapter 2: Sets, Functions, Sequences, and Sums

§2.1: Sets

Introduction to Set Theory

- A ***set*** is a new type of structure, representing an *unordered* collection (group, plurality) of zero or more *distinct* (different) objects.
- Set theory deals with operations between, relations among, and statements about sets.
- Sets are ubiquitous in computer software systems.
- *All* of mathematics can be defined in terms of some form of set theory (using predicate logic).

Basic notations for sets

- For sets, we'll use variables *S*, *T*, *U*, …
- We can denote a set *S* in writing by listing all of its elements in curly braces:
  - {a, b, c} is the set of whatever 3 objects are denoted by a, b, c.
- *Set builder notation*: For any proposition $P(x)$ over any universe of discourse, $\{x|P(x)\}$ is *the set of all x such that* $P(x)$.

Basic properties of sets

- Sets are inherently *unordered*:
  - No matter what objects a, b, and c denote, {a, b, c} = {a, c, b} = {b, a, c} = {b, c, a} = {c, a, b} = {c, b, a}.
- All elements are *distinct* (unequal); multiple listings make no difference!
  - If a=b, then {a, b, c} = {a, c} = {b, c} = {a, a, b, a, b, c, c, c, c}.
  - This set contains at most 2 elements!

Definition of Set Equality

- Two sets are declared to be equal *if and only if* they contain exactly the same elements.
- In particular, it does not matter *how the set is defined or denoted.*
- For example: The set {1, 2, 3, 4} =
  {x | x is an integer where x>0 and x<5 } =
  {x | x is a positive integer whose square is >0 and <25}

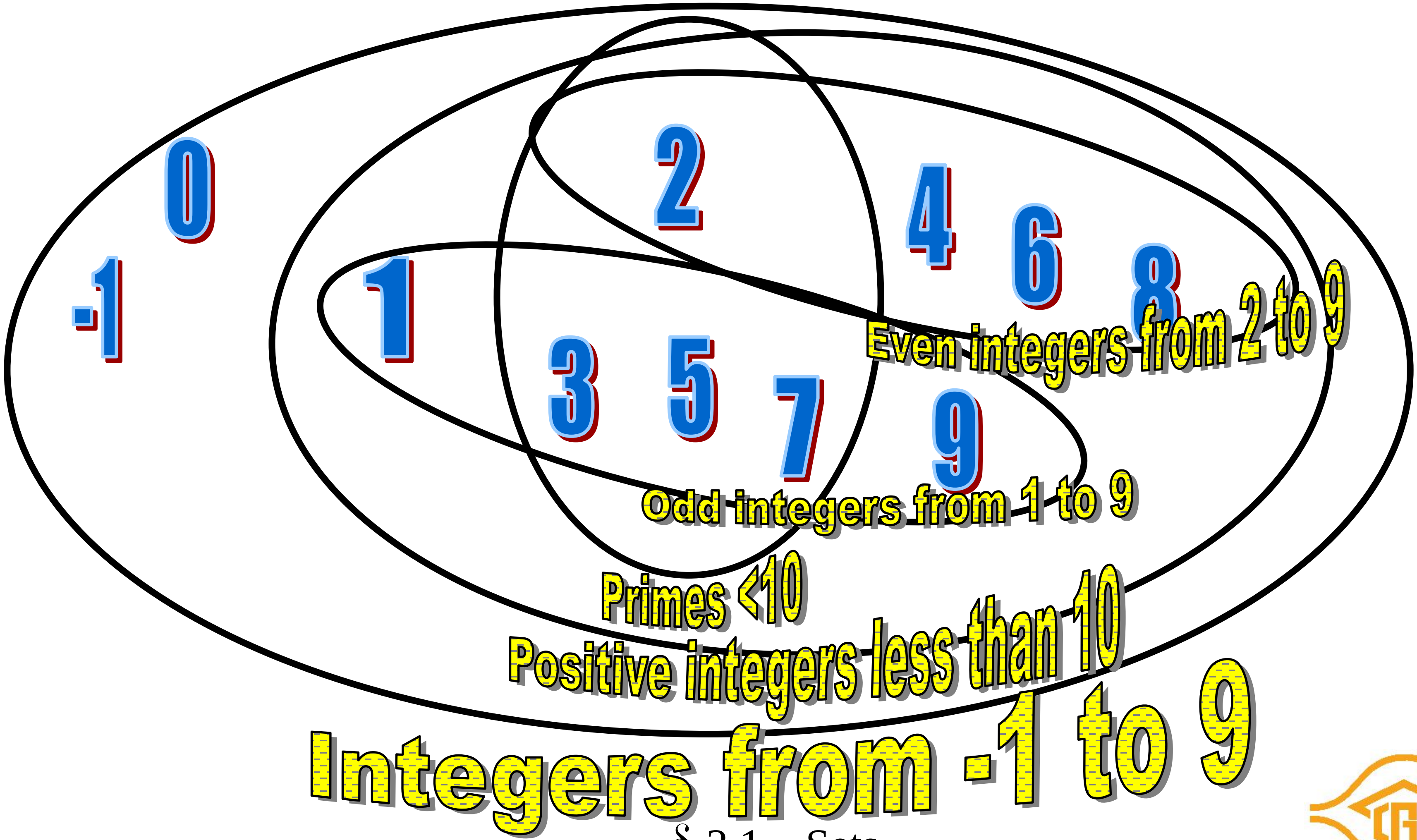
Infinite Sets

- Conceptually, sets may be *infinite* (*i.e.*, not *finite*, without end, unending).
- Symbols for some special infinite sets:

  N = {0, 1, 2, …} The **N**atural numbers.
  Z = {…, -2, -1, 0, 1, 2, …} The *Z*ntegers.

  R = The "**R**eal" numbers, such as 374.1828471929498181917281943125…
- Infinite sets come in different sizes!

Venn Diagrams



Basic Set Relations: Member of

- $x \in S$ ("x is in S") is the proposition that object x is an $\in$*lement* or *member* of set S.
  - *e.g.* $3 \in \mathbf{N}$, "a" $\in \{x \mid x$ is a letter of the alphabet$\}$
  - Can define set equality in terms of $\in$ relation: $\forall S,T$: $S=T \leftrightarrow (\forall x: x \in S \leftrightarrow x \in T)$ "Two sets are equal iff they have all the same members."

The Empty Set

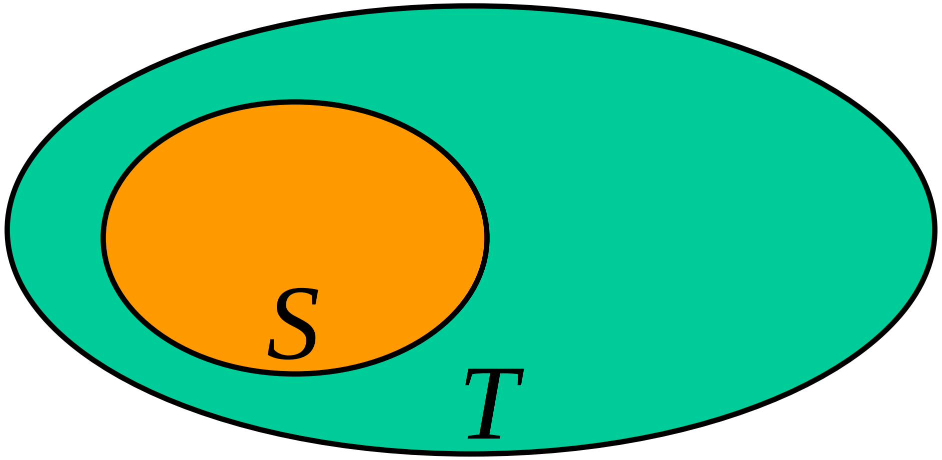
- $\varnothing$ ("null", "the empty set") is the unique set that contains no elements whatsoever.
- $\varnothing = \{\} = \{x | \textbf{False}\}$
- No matter the domain of discourse, we have the axiom $\neg\exists x: x \in \varnothing$.

Subset and Superset Relations

- $S \subseteq T$ ("S is a subset of T") means that every element of S is also an element of T.
- $S \subseteq T \Leftrightarrow \forall x\, (x \in S \rightarrow x \in T)$
- $\emptyset \subseteq S$, $S \subseteq S$.
- $S \supseteq T$ ("S is a superset of T") means $T \subseteq S$.
- Note $S = T \Leftrightarrow S \subseteq T \wedge S \supseteq T$.
- $S \not\subseteq T$ means $\neg(S \subseteq T)$, *i.e.* $\exists x(x \in S \wedge x \notin T)$

Proper (Strict) Subsets & Supersets

- $S \subset T$ ("S is a proper subset of T") means that $S \subseteq T$ but $T \not\subseteq S$. Similar for $S \supset T$.



Example: $\{1,2\} \subset \{1,2,3\}$

Venn Diagram equivalent of $S \subset T$

Sets Are Objects, Too!

- The objects that are elements of a set may *themselves* be sets.
- *E.g.* let $S=\{x \mid x \subseteq \{1,2,3\}\}$
  then $S=\{$

  $\}$
- Note that $1 \neq \{1\} \neq \{\{1\}\}$!!!!

Very Important!

Cardinality and Finiteness

- $|S|$ (read "the *cardinality* of *S*") is a measure of how many different elements *S* has.
- *E.g.*, $|\varnothing|$= __, $|\{1,2,3\}|$ =___, $|\{a,b\}|$ =___,
  $|\{\{1,2,3\},\{4,5\}\}|$ = ____
- If $|S| \in \mathbf{N}$, then we say *S* is *finite*.
  Otherwise, we say *S* is *infinite*.
- What are some infinite sets we've seen?

N Z R

The *Power Set* Operation

- The *power set* P(S) of a set S is the set of all subsets of S. P(S) = $\{x \mid x \subseteq S\}$.
- *E.g.* P({a,b}) = { };.
- Sometimes P(S) is written **2**^{S}.
  Note that for finite S, $|P(S)| = 2^{|S|}$.
- It turns out that $|P(\mathbf{N})| > |\mathbf{N}|$.
  There are different sizes of infinite sets!

Review: Set Notations So Far

- Variable objects x, y, z; sets S, T, U.
- Literal set {a, b, c} and set-builder $\{x|P(x)\}$.
- $\in$ relational operator, and the empty set $\varnothing$.
- Set relations $=$, $\subseteq$, $\supseteq$, $\subset$, $\supset$, $\not\subset$, etc.
- Venn diagrams.
- Cardinality $|S|$ and infinite sets **N**, **Z**, **R**.
- Power sets P(S).

Ordered *n*-tuples

- These are like sets, except that duplicates matter, and the order makes a difference.
- For $n \in \mathbf{N}$, an *ordered n-tuple* or a *sequence of length n* is written $(a_1, a_2, \ldots, a_n)$. The *first* element is a_1, *etc.*
- Note $(1, 2) \neq (2, 1) \neq (2, 1, 1)$.
- Empty sequence, singlets, pairs, triples, quadruples, quintuples, …, *n*-tuples.

Cartesian Products of Sets

- For sets A, B, their *Cartesian product* $A \times B :\equiv \{(a, b) \mid a \in A \wedge b \in B\}$.
- *E.g.* {a,b}×{1,2} = { }
- Note that for finite A, B, $|A \times B| = |A||B|$.
- Note that the Cartesian product is *not* commutative: $\neg \forall AB: A \times B = B \times A$.
- Extends to $A_1 \times A_2 \times \ldots \times A_n$...

René Descartes (1596-1650)

Review of §2.1

- Sets S, T, U… Special sets **N**, **Z**, **R**.
- Set notations {a,b,...}, $\{x|P(x)\}$…
- Set relation operators $x\in S$, $S\subseteq T$, $S\supseteq T$, $S=T$, $S\subset T$, $S\supset T$. (These form propositions.)
- Finite vs. infinite sets.
- Set operations $|S|$, P(S), $S\times T$.
- Next up: §2.2: More set ops: $\cup$, $\cap$, $-$.

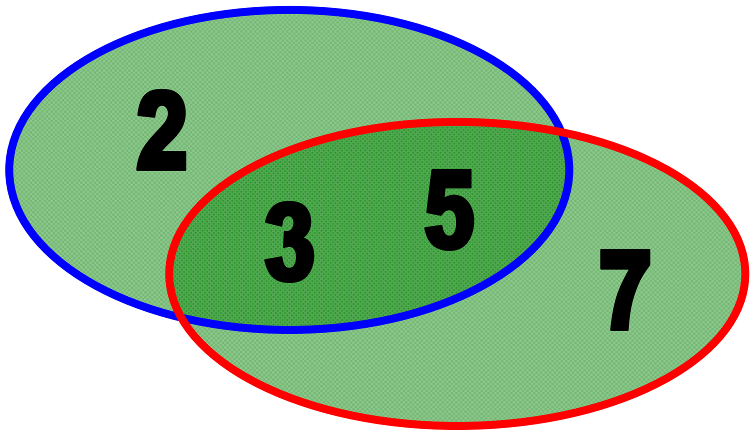
§2.2: Set Operations

The Union Operator

- For sets A, B, their $\cup$*nion* $A\cup B$ is the set containing all elements that are either in A, **or** ("$\vee$") in B (or, of course, in both).
- Formally, $\forall A,B$: $A\cup B = \{x \mid x\in A \vee x\in B\}$.
- Note that $A\cup B$ contains all the elements of A **and** it contains all the elements of B: $\forall A, B$: $(A\cup B \supseteq A) \wedge (A\cup B \supseteq B)$

Union Examples

- $\{a,b,c\} \cup \{2,3\} = \{ \qquad \}$ Required Form
- $\{2,3,5\} \cup \{3,5,7\} = \{2,3,5,3,5,7\} = \{ \qquad \}$



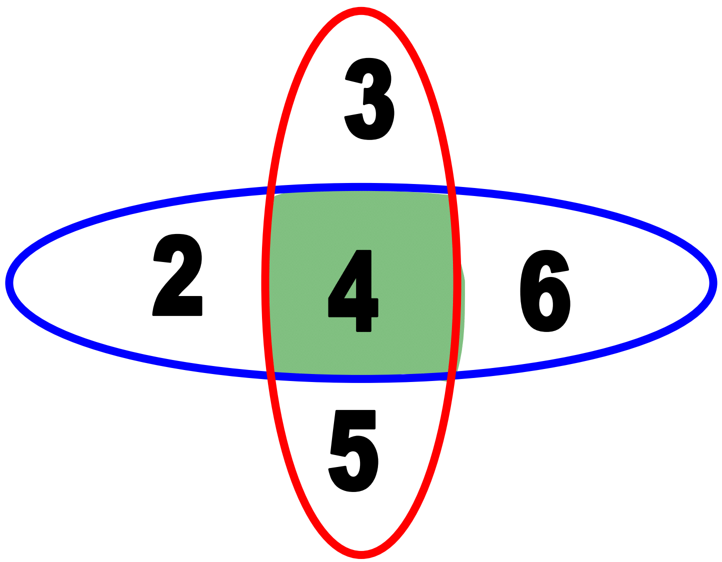
Think "The **Uni**ted States of America includes every person who worked in any U.S. state last year." (This is how the IRS sees it...)

The Intersection Operator

- For sets A, B, their *intersection* $A\cap B$ is the set containing all elements that are simultaneously in A **and** (“$\wedge$”) in B.
- Formally, $\forall A,B$: $A\cap B \equiv \{x \mid x\in A \wedge x\in B\}$.
- Note that $A\cap B$ is a subset of A **and** it is a subset of B:
  $\forall A, B$: $(A\cap B \subseteq A) \wedge (A\cap B \subseteq B)$

Intersection Examples

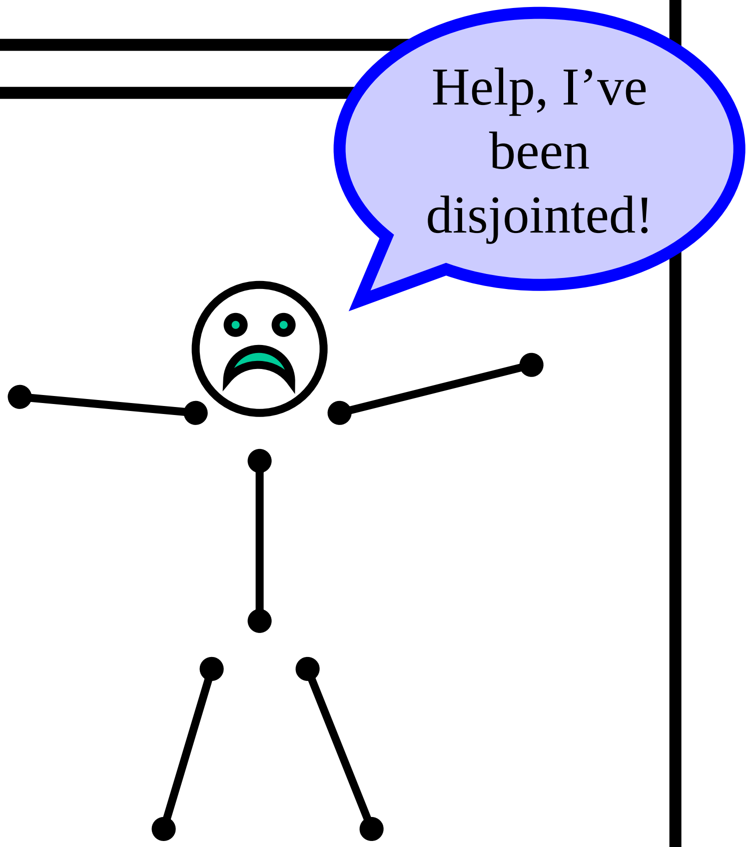
- $\{a,b,c\}\cap\{2,3\}$ = ___
- $\{2,4,6\}\cap\{3,4,5\}$ = ______



Think "The **intersection** of University Ave. and W 13th St. is just that part of the road surface that lies on *both* streets."

Disjointedness

- Two sets *A*, *B* are called ***disjoint*** (*i.e.*, unjoined) iff their intersection is empty. ($A\cap B=\varnothing$)
- Example: the set of even integers is disjoint with the set of odd integers.



Inclusion-Exclusion Principle

- How many elements are in $A \cup B$?
  $|A \cup B| =$
- Example: How many students are on our class email list? Consider set $E = I \cup M$,
  $I = \{s \mid s \text{ turned in an information sheet}\}$
  $M = \{s \mid s \text{ sent the TAs their email address}\}$
- Some students did both!
  $|E| = |I \cup M| = |I| + |M| - |I \cap M|$

Set Difference

- For sets *A*, *B*, the *difference of A and B*, written *A–B*, is the set of all elements that are in *A* but not *B*.
- $A - B :\equiv \{x \mid x \in A \wedge x \notin B\}$
  $= \{x \mid \neg(x \in A \rightarrow x \in B)\}$
- Also called:
  The *complement of B with respect to A*.

Set Difference Examples

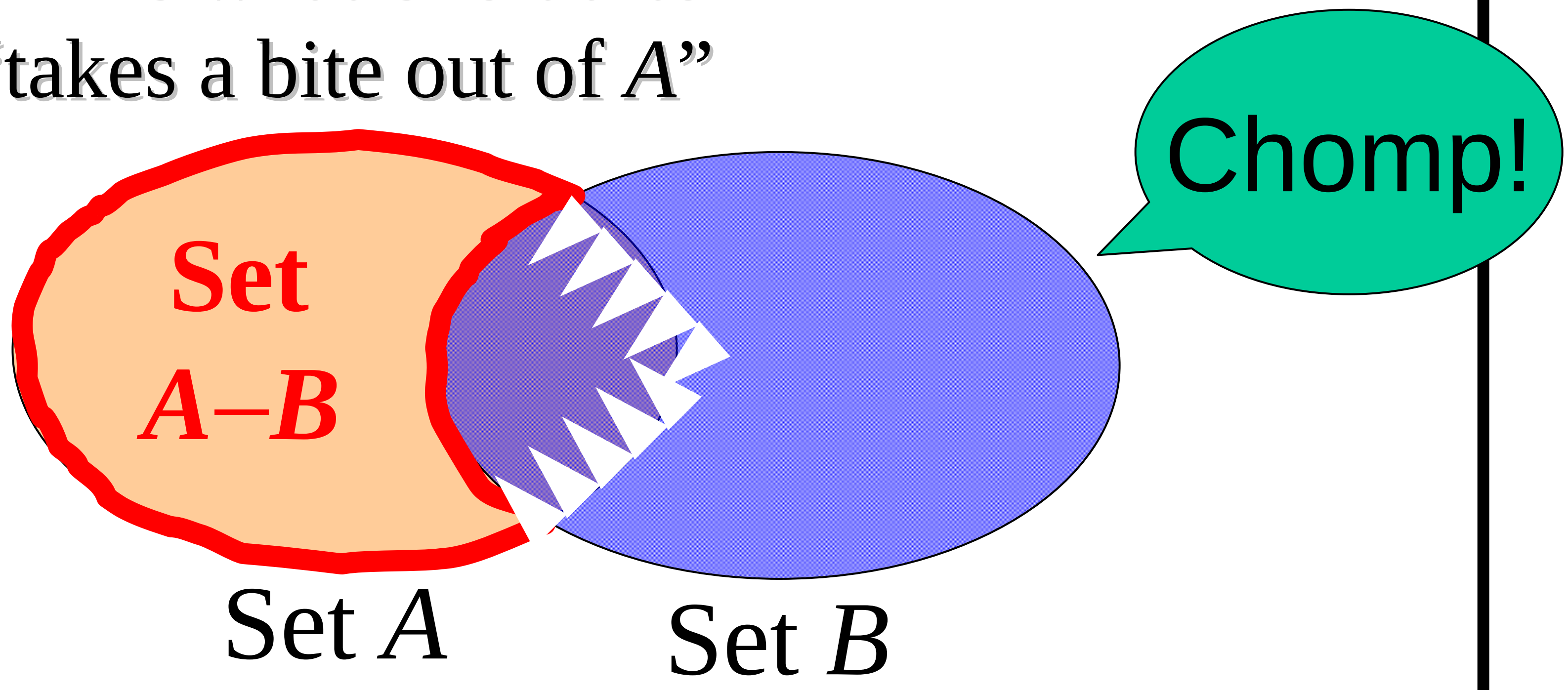
- {1,2,3,4,5,6} – {2,3,5,7,9,11} =

  ____

- **Z** – **N** = {… , -1, 0, 1, 2, … } – {0, 1, … }
  = {*x* | *x* is an integer but not a nat. #}
  = {*x* | *x* is a negative integer}
  = {… , -3, -2, -1}

Set Difference - Venn Diagram

- *A*-*B* is what's left after *B* "takes a bite out of *A*"



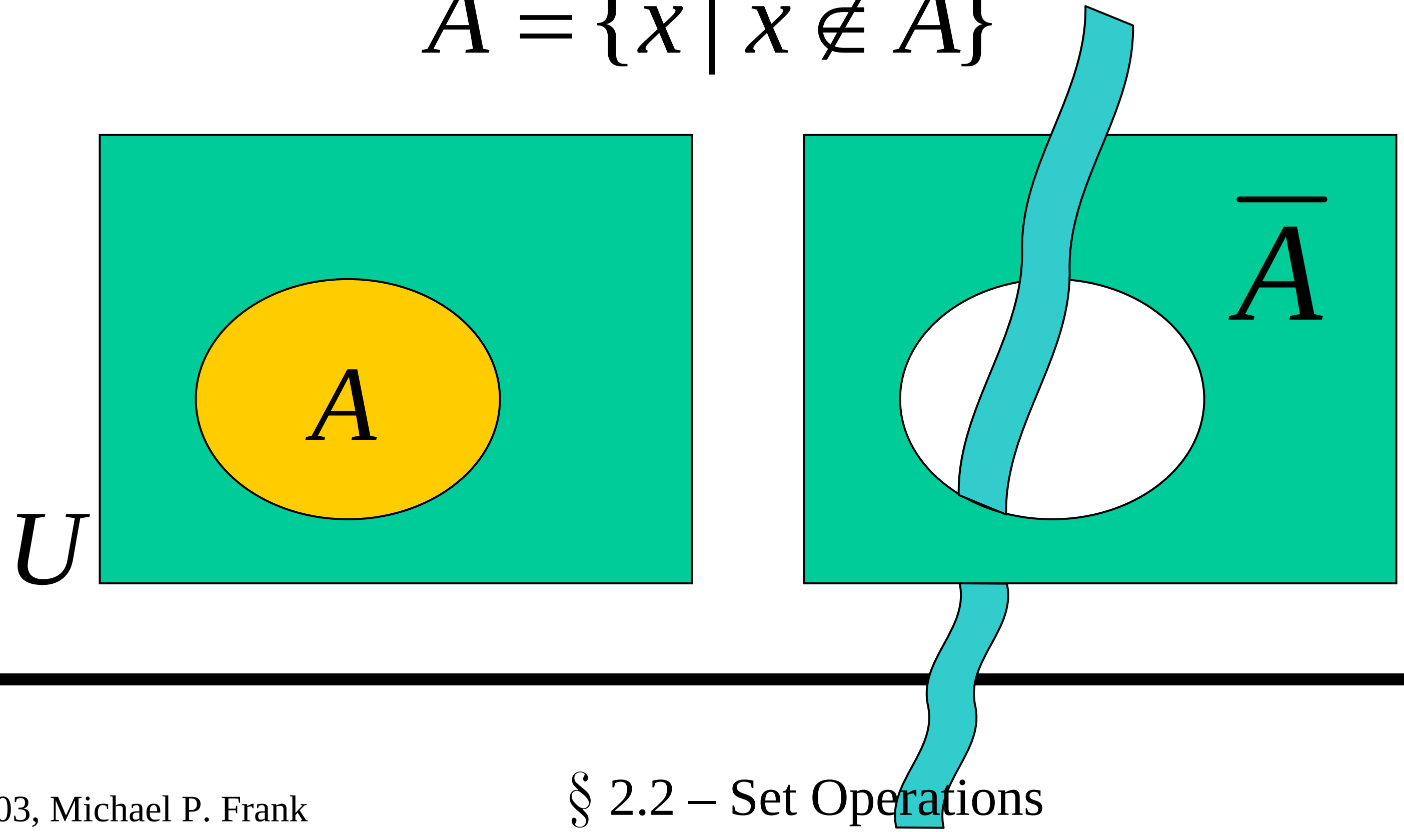
Set Complements

- The *universe of discourse* can itself be considered a set, call it U.
- When the context clearly defines U, we say that for any set $A \subseteq U$, the *complement* of A, written $\overline{A}$, is the complement of A w.r.t. U, *i.e.*, it is $U-A$.
- *E.g.*, If $U=\mathbf{N}$, $\overline{\{3,5\}} = \{0,1,2,4,6,7,\ldots\}$

More on Set Complements

- An equivalent definition, when *U* is clear:

$$\overline{A} = \{x \mid x \notin A\}$$



Set Identities

- Identity: $A \cup \varnothing = A \quad A \cap U = A$
- Domination: $A \cup U = U \quad A \cap \varnothing = \varnothing$
- Idempotent: $A \cup A = A = A \cap A$
- Double complement: $\overline{(\overline{A})} = A$
- Commutative: $A \cup B = B \cup A \quad A \cap B = B \cap A$
- Associative: $A \cup (B \cup C) = (A \cup B) \cup C$
  $A \cap (B \cap C) = (A \cap B) \cap C$

DeMorgan's Law for Sets

- Exactly analogous to (and derivable from) DeMorgan's Law for propositions.

$$\overline{A \cup B} =$$

$$\overline{A \cap B} =$$

Proving Set Identities

To prove statements about sets, of the form $E_1 = E_2$ (where Es are set expressions), here are three useful techniques:

- Prove $E_1 \subseteq E_2$ and $E_2 \subseteq E_1$ separately.
- Use set builder notation & logical equivalences.
- Use a *membership table*.

Method 1: Mutual subsets

Example: Show $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$.

- Show $A\cap(B\cup C)\subseteq(A\cap B)\cup(A\cap C)$.
  - Assume $x\in A\cap(B\cup C)$, & show $x\in(A\cap B)\cup(A\cap C)$.
  - We know that $x\in A$, and either $x\in B$ or $x\in C$.
    - Case 1: $x\in B$. Then $x\in A\cap B$, so $x\in(A\cap B)\cup(A\cap C)$.
    - Case 2: $x\in C$. Then $x\in A\cap C$, so $x\in(A\cap B)\cup(A\cap C)$.
  - Therefore, $x\in(A\cap B)\cup(A\cap C)$.
  - Therefore, $A\cap(B\cup C)\subseteq(A\cap B)\cup(A\cap C)$.
- Show $(A\cap B)\cup(A\cap C) \subseteq A\cap(B\cup C)$. …

Method 3: Membership Tables

- Just like truth tables for propositional logic.
- Columns for different set expressions.
- Rows for all combinations of memberships in constituent sets.
- Use "1" to indicate membership in the derived set, "0" for non-membership.
- Prove equivalence with identical columns.

Membership Table Example

Prove $(A \cup B) - B = A - B$.

A	B	$A \cup B$	$(A \cup B) - B$	$A - B$
0	0			
0	1			
1	0			
1	1			

Membership Table Exercise

Prove $(A \cup B) - C = (A - C) \cup (B - C)$.

A	B	C	$A \cup B$	$(A \cup B) - C$	$A - C$	$B - C$	$(A - C) \cup (B - C)$
0	0	0					
0	0	1					
0	1	0					
0	1	1					
1	0	0					
1	0	1					
1	1	0					
1	1	1					

Review of §2.1~2.2

- Sets S, T, U… Special sets **N**, **Z**, **R**.
- Set notations {a,b,...}, $\{x|P(x)\}$…
- Relations $x\in S$, $S\subseteq T$, $S\supseteq T$, $S=T$, $S\subset T$, $S\supset T$.
- Operations $|S|$, P(S), $\times$, $\cup$, $\cap$, $-$, $\overline{S}$
- Set equality proof techniques:
  – Mutual subsets.
  – Derivation using logical equivalences.

Generalized Unions & Intersections

- Since union & intersection are commutative and associative, we can extend them from operating on *ordered pairs* of sets (A,B) to operating on sequences of sets $(A_1,\ldots,A_n)$, or even unordered *sets* of sets, $X=\{A \mid P(A)\}$.

Generalized Union

- Binary union operator: $A \cup B$
- n-ary union:
  $A \cup A_2 \cup \ldots \cup A_n :\equiv ((\ldots((A_1 \cup A_2) \cup \ldots) \cup A_n)$
  (grouping & order is irrelevant)
- "Big U" notation: $\bigcup_{i=1}^{n} A_i$
- Or for infinite sets of sets: $\bigcup_{A \in X} A$

Generalized Intersection

- Binary intersection operator: $A \cap B$
- n-ary intersection:
  $A \cap A_2 \cap \ldots \cap A_n \equiv ((\ldots((A_1 \cap A_2) \cap \ldots) \cap A_n)$
  (grouping & order is irrelevant)
- "Big Arch" notation: $\bigcap_{i=1}^{n} A_i$
- Or for infinite sets of sets: $\bigcap_{A \in X} A$

Representations

- A frequent theme of this course will be methods of *representing* one discrete structure using another discrete structure of a different type.
- *E.g.*, one can represent natural numbers as
  - Sets: **0**:≡∅, **1**:≡{**0**}, **2**:≡{**0**,**1**}, **3**:≡{**0**,**1**,**2**}, …
  - Bit strings:
    0:≡`0`, **1**:≡`1`, **2**:≡`10`, **3**:≡`11`, **4**:≡`100`, …

Representing Sets with Bit Strings

For an enumerable u.d. U with ordering $\{x_1, x_2, \ldots\}$, represent a finite set $S \subseteq U$ as the finite bit string B=$b_1 b_2 \ldots b_n$ where $\forall i$: $x_i \in S \leftrightarrow (i<n \wedge b_i=1)$.

E.g. $U=\mathbf{N}$, $S=\{2,3,5,7,11\}$, B=001101010001.

In this representation, the set operators "$\cup$", "$\cap$", "$\overline{\ \ }$" are implemented directly by bitwise OR, AND, NOT!

Symmetric Difference of Sets

Symmetric Difference of A and B, denoted as $A \oplus B$, where

$A \oplus B = \{x \mid x \text{ in A or in B, but not both}\}$.

E.g: $A \oplus B = (A \cup B) - (A \cap B)$

$= (A - B) \cup (B - A)$

Do it in homework!

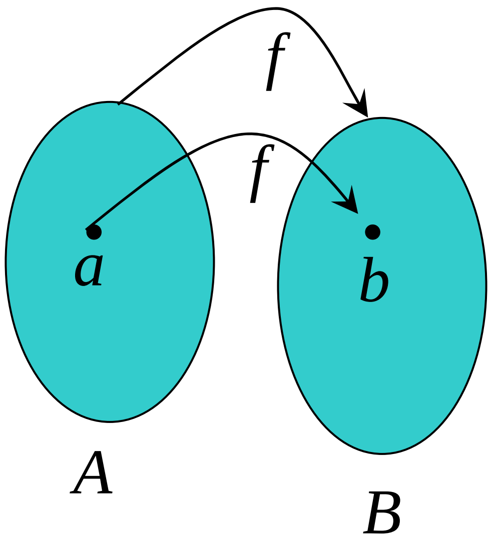
§2.3: Functions

Function: Formal Definition

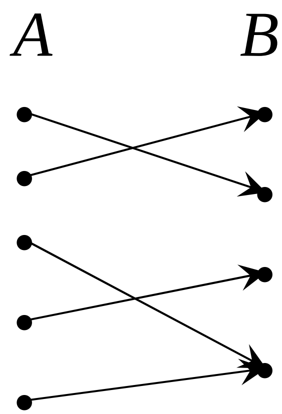
- For any sets A, B, we say that a *function f from (or “mapping”) A to B* (f:$A \rightarrow B$) is a particular assignment of exactly one element $f(x) \in B$ to each element $x \in A$.
- Some further generalizations of this idea:
  - A *partial* (non-*total*) function f assigns *zero or one* elements of B to each element $x \in A$.
  - Functions of n arguments; relations (ch. 8).

Graphical Representations

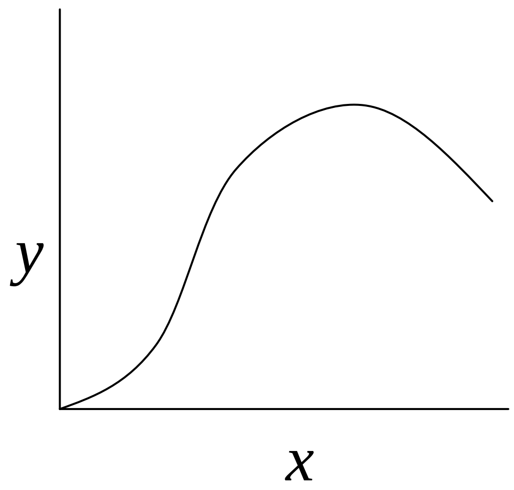
- Functions can be represented graphically in several ways:



Like Venn diagrams



Bipartite Graph



Plot

Functions We've Seen So Far

- A *proposition* can be viewed as a function from "situations" to truth values {**T**,**F**}
  - A logic system called *situation theory*.
  - *p*="It is raining."; *s*=our situation here,now
  - $p(s) \in \{\mathbf{T},\mathbf{F}\}$.
- A *propositional operator* can be viewed as a function from *ordered pairs* of truth values to truth values: $\vee((\mathbf{F},\mathbf{T})) = \mathbf{T}$.

Another example: $\rightarrow((\mathbf{T},\mathbf{F})) = \mathbf{F}$.

More functions so far…

- A *predicate* can be viewed as a function from *objects* to *propositions* (or truth values): $P :\equiv$ "is 7 feet tall"; *P*(Mike) = "Mike is 7 feet tall." = **False**.
- A *bit string B of length n* can be viewed as a function from the numbers {1,…,*n*} (bit *positions*) to the *bits* {`0`,`1`}. *E.g.*, *B*=`101` ➔ *B*(3)=____.

Still More Functions

- A *set S* over universe *U* can be viewed as a function from the elements of *U* to {**T**, **F**}, saying for each element of *U* whether it is in *S*. *S*={3}; *S*(0)=**F**, S(3)=**T**.
- A *set operator* such as $\cap$, $\cup$, $\overline{\ }$ can be viewed as a function from pairs of sets to sets.
  - Example: $\cap((\{1,3\},\{3,4\}))$ =

A Neat Trick

- Sometimes we write Y^X to denote the set F of *all* possible functions $f: X \rightarrow Y$.
- This notation is especially appropriate, because for finite X, Y, $|F| = |Y|^{|X|}$.
- If we use representations $\mathbf{F} \equiv \mathbf{0}$, $\mathbf{T} \equiv \mathbf{1}$, $\mathbf{2} :\equiv \{\mathbf{0},\mathbf{1}\} = \{\mathbf{F},\mathbf{T}\}$, then a subset $T \subseteq S$ is just a function from S to $\mathbf{2}$, so the power set of S (set of all such fns.) is $\mathbf{2}^S$ in this notation.

Some Function Terminology

- If f:$A \rightarrow B$, and $f(a)=b$ (where $a \in A$ & $b \in B$), then:
  - – A is the *domain* of f.
  - – B is the *codomain* of f.
  - – b is the *image* of a under f.
  - – a is a *pre-image* of b under f.
    - In general, b may have more than 1 pre-image.
  - – The *range* $R \subseteq B$ of f is $\{b \mid \exists a\ f(a)=b\}$.

Range versus Codomain

- The range of a function might *not* be its whole codomain.
- The codomain is the set that the function is *declared* to map all domain values into.
- The range is the *particular* set of values in the codomain that the function *actually* maps elements of the domain to.

Range vs. Codomain - Example

- Suppose I declare to you that: "*f* is a function mapping students in this class to the set of grades {A,B,C,D,E}."
- At this point, you know *f*'s codomain is: __________, and its range is ________.
- Suppose the grades turn out all As and Bs.
- Then the range of *f* is _________, but its codomain is ___________________.

Operators (general definition)

- An n-ary *operator* over the set S is any function from the set of ordered n-tuples of elements of S, to S itself.
- *E.g.*, if $S=\{\mathbf{T},\mathbf{F}\}$, $\neg$ can be seen as a unary operator, and $\wedge,\vee$ are binary operators on S.
- Another example: $\cup$ and $\cap$ are binary operators on the set of all sets.

Constructing Function Operators

- If • ("dot") is any operator over B, then we can extend • to also denote an operator over functions $f : A \rightarrow B$.
- *E.g.*: Given any binary operator $\bullet: B \times B \rightarrow B$, and functions $f, g : A \rightarrow B$, we define $(f \bullet g): A \rightarrow B$ to be the function defined by: $\forall a \in A, (f \bullet g)(a) = f(a) \bullet g(a)$.

Function Operator Example

- $+, \times$ ("plus","times") are binary operators over **R**. (Normal addition & multiplication.)
- Therefore, we can also add and multiply *functions* $f, g : \mathbf{R} \rightarrow \mathbf{R}$:
  - $(f + g) : \mathbf{R} \rightarrow \mathbf{R}$, where $(f + g)(x) = f(x) + g(x)$
  - $(f \times g) : \mathbf{R} \rightarrow \mathbf{R}$, where $(f \times g)(x) = f(x) \times g(x)$

Function Composition Operator

- For functions $g:A\rightarrow B$ and $f:B\rightarrow C$, there is a special operator called *compose* ("o").
  - It composes (creates) a new function out of f,g by applying f to the result of g.
  - $(f \circ g) : A\rightarrow C$, where $(f \circ g)(a) = f(g(a))$.
  - Note $g(a)\in B$, so $f(g(a))$ is defined and $\in C$.
  - Note that o (like Cartesian $\times$, but unlike $+,\wedge,\cup$) is non-commuting. (Generally, $f \circ g \neq g \circ f$.)

Images of Sets under Functions

- Given $f: A \to B$, and $S \subseteq A$,
- The *image of S* under f is simply the set of all images (under f) of the elements of S.
  $f(S) :\equiv \{f(s) \mid s \in S\}$
  $:\equiv \{b \mid \exists\, s \in S: f(s) = b\}$.
- Note the range of f can be defined as simply the image (under f) of f's domain!

One-to-One Functions

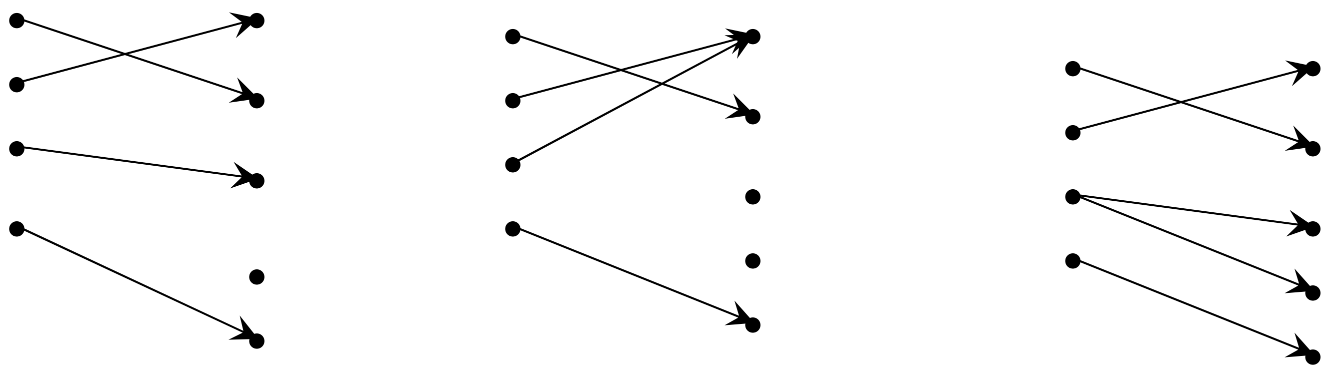
- A function is *one-to-one (1-1)*, or *injective*, or *an injection*, iff every element of its range has *only* 1 pre-image.
  - Formally: given $f: A \rightarrow B$,
    "x is injective" $:\equiv (\neg\exists x,y\colon x \neq y \wedge f(x)=f(y))$.
- Only one element of the domain is mapped to any given one element of the range.
  - Domain & range have same cardinality. What about codomain?

May Be Larger

- Each element of the domain is injected into a different element of the range.
  - Compare "each dose of vaccine is injected into a different patient."

One-to-One Illustration

- Bipartite (2-part) graph representations of functions that are (or not) one-to-one:



One-to-one

Not one-to-one

Not even a function!

Sufficient Conditions for 1-1ness

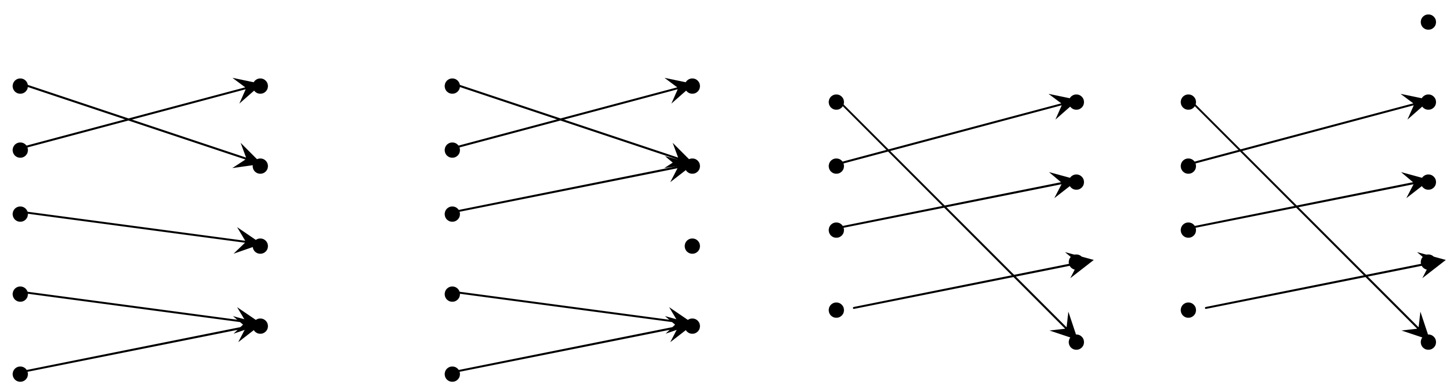
- For functions *f* over numbers,
  - *f* is *strictly* (or *monotonically*) *increasing* iff $x>y \rightarrow f(x)>f(y)$ for all *x*,*y* in domain;
  - *f* is *strictly* (or *monotonically*) *decreasing* iff $x>y \rightarrow f(x)<f(y)$ for all *x*,*y* in domain;
- If *f* is either strictly increasing or strictly decreasing, then *f* is one-to-one. *E.g.* x^3
  - *Converse is not necessarily true*. *E.g.* $1/x$

Onto (Surjective) Functions

- A function $f : A \rightarrow B$ is *onto* or *surjective* or *a surjection* iff its range is equal to its codomain ($\forall b \in B, \exists a \in A: f(a)=b$).
- An *onto* function maps the set *A* onto (over, covering) the *entirety* of the set *B*, not just over a piece of it.
- *E.g.*, for domain & codomain **R**, x^3 is onto, whereas x^2 isn't. (Why not?)

Illustration of Onto

- Some functions that are or are not *onto* their codomains:



Onto (but not 1-1) | Not Onto (or 1-1) | Both 1-1 and onto | 1-1 but not onto

Bijections

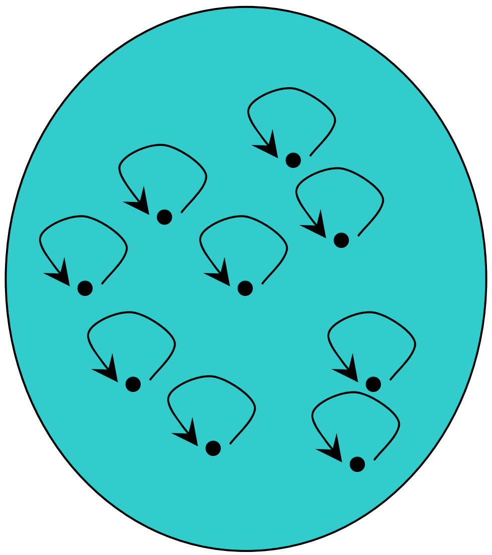
- A function f is *a one-to-one correspondence*, or *a bijection*, or *reversible*, or *invertible*, iff it is both one-to-one and onto.
- For bijections $f: A \rightarrow B$, there exists an *inverse* of f, written $f^{-1}: B \rightarrow A$, which is the unique function such that $f^{-1} \circ f = I$ (the identity function)

The Identity Function

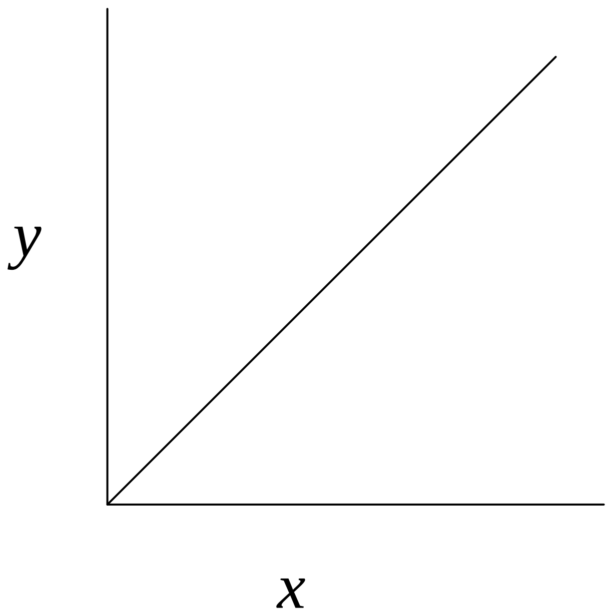
- For any domain A, the *identity function* $I:A\rightarrow A$ (variously written, I_A, **1**, $\mathbf{1}_A$) is the unique function such that $\forall a\in A$: $I(a)=a$.
- Some identity functions you've seen:
  - $+$ing 0, $\cdot$ing by 1, $\wedge$ing with **T**, $\vee$ing with **F**, $\cup$ing with $\varnothing$, $\cap$ing with U.
- Note that the identity function is both one-to-one and onto (bijective).

Identity Function Illustrations

- The identity function:



Domain and range



Graphs of Functions

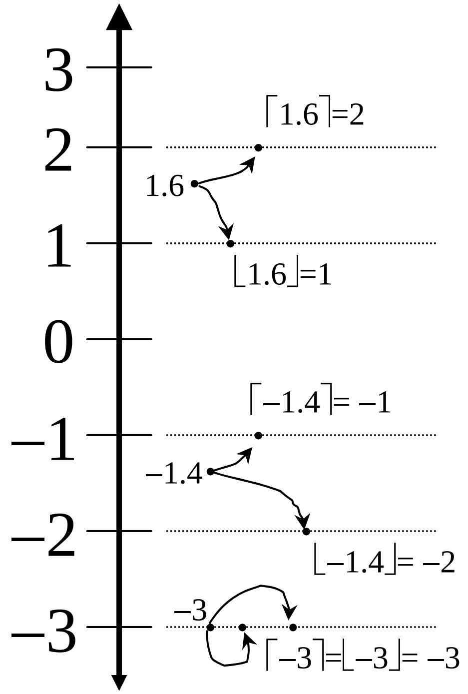
- We can represent a function $f : A \rightarrow B$ as a set of ordered pairs $\{(a, f(a)) \mid a \in A\}$.
- Note that $\forall a$, there is only 1 pair $(a, f(a))$.
  - Later (ch.8): *relations* loosen this restriction.
- For functions over numbers, we can represent an ordered pair (x, y) as a point on a plane. A function is then drawn as a curve (set of points) with only one y for each x.

A Couple of Key Functions

- In discrete math, we will frequently use the following functions over real numbers:
  - $\lfloor x \rfloor$ ("floor of *x*") is the largest (most positive) integer $\leq x$.
  - $\lceil x \rceil$ ("ceiling of *x*") is the smallest (most negative) integer $\geq x$.

Visualizing Floor & Ceiling

- Real numbers "fall to their floor" or "rise to their ceiling."
- Note that if $x \notin \mathbf{Z}$, $\lfloor -x \rfloor \neq -\lfloor x \rfloor$ & $\lceil -x \rceil \neq -\lceil x \rceil$
- Note that if $x \in \mathbf{Z}$, $\lfloor x \rfloor = \lceil x \rceil = x$.

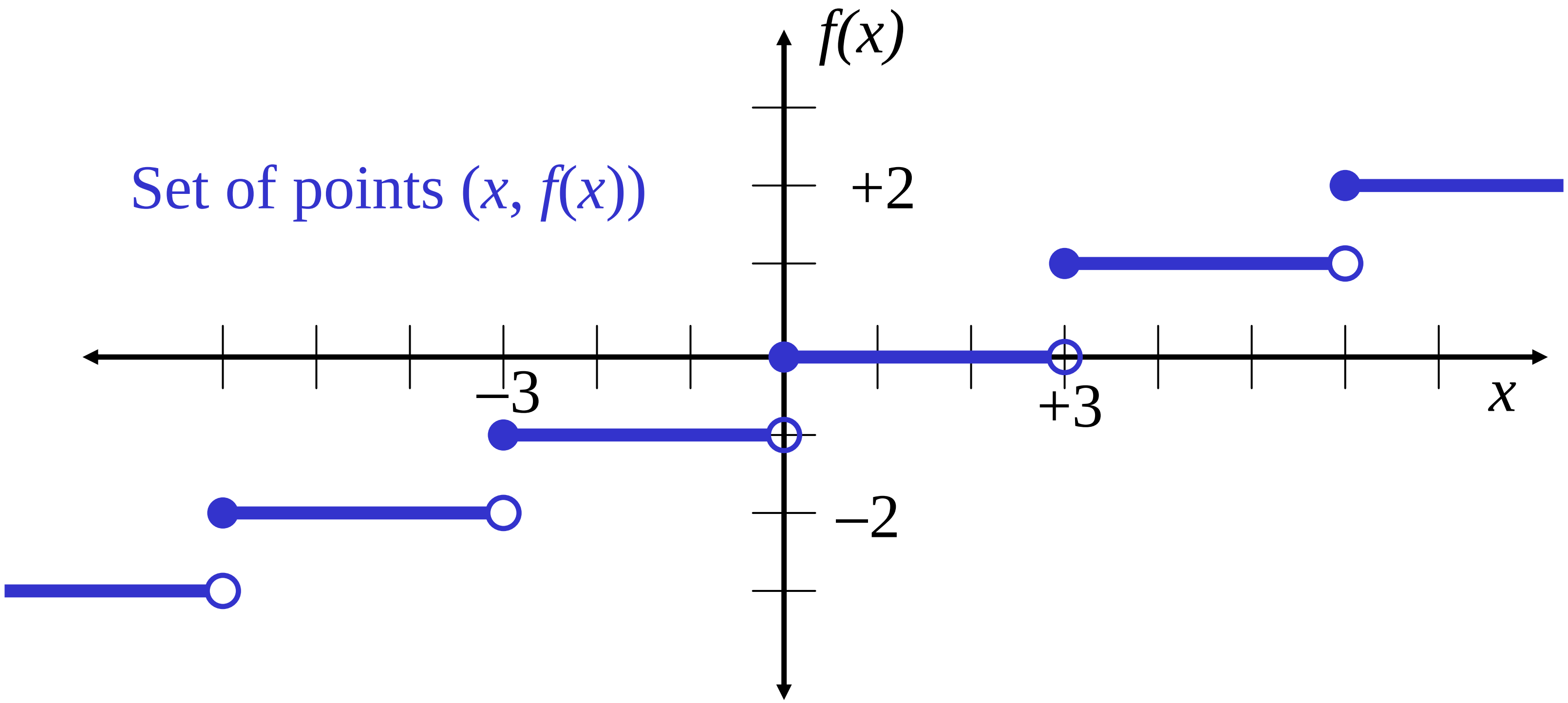


Plots with floor/ceiling

Note that for $f(x)=\lfloor x \rfloor$, the graph of f includes the point $(a, 0)$ for all values of a such that $a \geq 0$ and $a<1$, but not for $a=1$. We say that the set of points $(a,0)$ that is in f does not include its *limit* or *boundary* point $(a,1)$. Sets that do not include all of their limit points are called *open sets*. In a plot, we draw a limit point of a curve using an open dot (circle) if the limit point is not on the curve, and with a closed (solid) dot if it is on the curve.

Plots with floor/ceiling: Example

- Plot of graph of function $f(x) = \lfloor x/3 \rfloor$:



Review of §2.3 (Functions)

- Function variables f, g, h, …
- Notations: $f : A \rightarrow B$, $f(a)$, $f(A)$.
- Terms: image, preimage, domain, codomain, range, one-to-one, onto, strictly (in/de)creasing, bijective, inverse, composition.
- Function unary operator f^{-1},
  binary operators +, −, *etc.*, and $\bigcirc$.
- The **R**→**Z** functions $\lfloor x \rfloor$ and $\lceil x \rceil$.

§ 2.4: Sequences and Summations

Sequences & Strings

- A *sequence* or *series* is just like an ordered *n*-tuple, except:
  - Each element in the series has an associated *index* number.
  - A sequence or series may be infinite.
- A *summation* is a compact notation for the sum of all terms in a (possibly infinite) series.

Sequences

- Formally: A ***sequence*** or ***series*** $\{a_n\}$ is identified with a *generating function* $f:S\rightarrow A$ for some subset $S\subseteq\mathbf{N}$ (often $S=\mathbf{N}$ or $S=\mathbf{N}-\{0\}$) and for some set A.
- If f is a generating function for a series $\{a_n\}$, then for $n\in S$, the symbol a_n denotes $f(n)$, also called *term n* of the sequence.
- The *index* of a_n is n. (Or, often i is used.)

Sequence Examples

- Many sources just write "the sequence a_1, a_2, …" instead of $\{a_n\}$, to ensure that the set of indices is clear.
  - Our book leaves it ambiguous.
- An example of an infinite series:
  - Consider the series $\{a_n\} = a_1, a_2, \ldots$, where $(\forall n \geq 1)\ a_n = f(n) = 1/n$.
  - Then $\{a_n\} = 1, 1/2, 1/3, \ldots$

Example with Repetitions

- Consider the sequence $\{b_n\} = b_0, b_1, \ldots$ (note 0 is an index) where $b_n = (-1)^n$.
- $\{b_n\} = 1, -1, 1, -1, \ldots$
- Note repetitions! $\{b_n\}$ denotes an infinite sequence of 1's and –1's, *not* the 2-element set $\{1, -1\}$.

Recognizing Sequences

- Sometimes, you're given the first few terms of a sequence, and you are asked to find the sequence's generating function, or a procedure to enumerate the sequence.
- Examples: What's the next number?
  - 1,2,3,4,… 5 (the 5th smallest number >0)
  - 1,3,5,7,9,… 11 (the 6th smallest odd number >0)
  - 2,3,5,7,11,... 13 (the 6th smallest prime number)

The Trouble with Recognition

- The problem of finding "the" generating function given just an initial subsequence is *not well defined*.
- This is because there are *infinitely* many computable functions that will generate *any* given initial subsequence.
- We implicitly are supposed to find the *simplest* such function (because this one is assumed to be most likely), but, how should we define the *simplicity* of a function?
  - We might define simplicity as the reciprocal of complexity, but…
  - There are *many* plausible, competing definitions of complexity, and this is an active research area.
- So, these questions really have *no* objective right answer!

What are Strings, Really?

- This book says "finite sequences of the form $a_1, a_2, \ldots, a_n$ are called *strings*", but *infinite* strings are also used sometimes.
- Strings are often restricted to sequences composed of *symbols* drawn from a finite *alphabet*, and may be indexed from 0 or 1.
- Either way, the length of a (finite) string is its number of terms (or of distinct indexes).

Strings, more formally

- Let Σ be a finite set of *symbols*, *i.e.* an *alphabet*.
- A *string s* over alphabet Σ is any sequence $\{s_i\}$ of symbols, $s_i \in \Sigma$, indexed by **N** or **N**–{0}.
- If *a*, *b*, *c*, … are symbols, the string *s* = *a*, *b*, *c*, … can also be written *abc* …(*i.e.*, without commas).
- If *s* is a finite string and *t* is a string, the *concatenation of s with t*, written *st*, is the string consisting of the symbols in *s*, in sequence, followed by the symbols in *t*, in sequence.

More String Notation

- The length $|s|$ of a finite string s is its number of *positions* (*i.e.*, its number of index values i).
- If s is a finite string and $n \in \mathbf{N}$, s^n denotes the concatenation of n copies of s.
- ε denotes the empty string, the string of length 0.
- If Σ is an alphabet and $n \in \mathbf{N}$,

  $\Sigma^n \equiv \{s \mid s \text{ is a string over } \Sigma \text{ of length } n\}$, and

  $\Sigma^* \equiv \{s \mid s \text{ is a finite string over } \Sigma\}$.

Summation Notation

- Given a series $\{a_n\}$, an integer *lower bound* (or *limit*) $j \geq 0$, and an integer *upper bound* $k \geq j$, then the *summation of* $\{a_n\}$ *from j to k* is written and defined as follows:

$$\sum_{i=j}^{k} a_i :\equiv a_j + a_{j+1} + \ldots + a_k$$

- Here, *i* is called the *index of summation*.

Generalized Summations

- For an infinite series, we may write:

$$\sum_{i=j}^{\infty} a_i :\equiv a_j + a_{j+1} + \ldots$$

- To sum a function over all members of a set $X=\{x_1, x_2, \ldots\}$: $\sum_{x \in X} f(x) :\equiv f(x_1) + f(x_2) + \ldots$
- Or, if $X=\{x|P(x)\}$, we may just write:

$$\sum_{P(x)} f(x) :\equiv f(x_1) + f(x_2) + \ldots$$

Simple Summation Example

$$\sum_{i=2}^{4} i^2 + 1 =$$

$$= (4+1) + (9+1) + (16+1)$$
$$= 5 + 10 + 17$$
$$= 32$$

More Summation Examples

- An infinite series with a finite sum:

$$\sum_{i=0}^{\infty} 2^{-i} =$$

- Using a predicate to define a set of elements to sum over:

$$\sum_{\substack{(x \text{ is prime}) \wedge \\ x<10}} x^2 =$$

Summation Manipulations

- Some handy identities for summations:

$$\sum_x cf(x) = c\sum_x f(x)$$ (Distributive law.)

$$\sum_x f(x) + g(x) = \left(\sum_x f(x)\right) + \sum_x g(x)$$ (Application of commut-ativity.)

$$\sum_{i=j}^{k} f(i) = \sum_{i=j+n}^{k+n} f(i-n)$$ (Index shifting.)

More Summation Manipulations

- Other identities that are sometimes useful:

$$\sum_{i=j}^{k} f(i) = \left(\sum_{i=j}^{m} f(i) \right) + \sum_{i=m+1}^{k} f(i) \quad \text{if } j \le m < k$$

(Series splitting.)

$$\sum_{i=j}^{k} f(i) = \sum_{i=0}^{k-j} f(k-i)$$

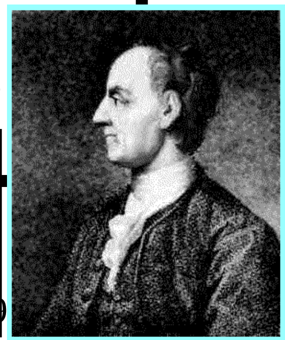
(Order reversal.)

$$\sum_{i=0}^{2k} f(i) = \left\{ \sum_{i=0}^{k} f(2i) + f(2i+1) \right\} - f(2k+1)$$

(Grouping.)

Example: Impress Your Friends

- Boast, "I'm so smart; give me any 2-digit number *n*, and I'll add all the numbers from 1 to *n* in my head in just a few seconds."
- *I.e.*, Evaluate the summation: $\sum_{i=1}^{n} i$
- There is a simple closed-form formula for the result, discovered by Euler at age 12!

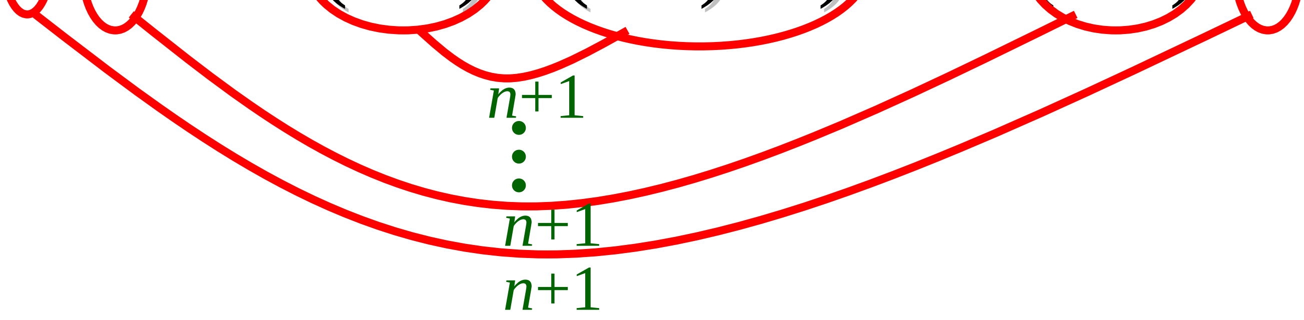


Leonhard Euler (1707-1783)

Euler's Trick, Illustrated

- Consider the sum:

$1+2+\ldots+(n/2)+((n/2)+1)+\ldots+(n-1)+n$



- $n/2$ pairs of elements, each pair summing to $n+1$, for a total of $(n/2)(n+1)$.

Symbolic Derivation of Trick

$$2\sum_{i=1}^{n} i = \left(\sum_{i=1}^{n} i + \sum_{i=1}^{n} i \right)$$

Concluding Euler's Derivation

$$\therefore \sum_{i=1}^{n} i =$$

- So, you only have to do 1 easy multiplication in your head, then cut in half.
- Also works for odd *n* (prove this at home).

Example: Geometric Progression

- A *geometric progression* is a series of the form $a, ar, ar^2, ar^3, \ldots, ar^k$, where $a,r \in \mathbf{R}$.
- The sum of such a series is given by:

$$S = \sum_{i=0}^{k} ar^i$$

- We can reduce this to *closed form* via clever manipulation of summations...

Geometric Sum Derivation

- Here we go...

$$S = \sum_{i=0}^{n} ar^i$$

$$rS = r\sum_{i=0}^{n} ar^i$$

Concluding long derivation…

$$\therefore rS = S + a(r^{n+1} - 1)$$

When $r = 1$, $S = \sum_{i=0}^{n} ar^i = \sum_{i=0}^{n} a1^i = \sum_{i=0}^{n} a \cdot 1 = (n+1)a$

Nested Summations

- These have the meaning you'd expect.

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}\left(\sum_{j=1}^{3} ij\right) = \sum_{i=1}^{4} i\left(\sum_{j=1}^{3} j\right) = \sum_{i=1}^{4} i(1+2+3)$$

$$= \sum_{i=1}^{4} 6i = 6\sum_{i=1}^{4} i = 6(1+2+3+4)$$

$$= 6 \cdot 10 = 60$$

- Note issues of free vs. bound variables, just like in quantified expressions, integrals, etc.

Some Shortcut Expressions

$$\sum_{k=0}^{n} ar^k = a(r^{n+1}-1)/(r-1), r \neq 1$$

Geometric series.

$$\sum_{k=1}^{n} k =$$

Euler's trick.

$$\sum_{k=1}^{n} k^2 =$$

Quadratic series.

$$\sum_{k=1}^{n} k^3 =$$

Cubic series.

Using the Shortcuts

- Example: Evaluate $\sum_{k=50}^{100} k^2$.
  - Use series splitting.
  - Solve for desired summation.
  - Apply quadratic series rule.
  - Evaluate.

$$\sum_{k=50}^{100} k^2 =$$

$$= \frac{100 \cdot 101 \cdot 201}{6} - \frac{49 \cdot 50 \cdot 99}{6}$$

$$= 338{,}350 - 40{,}425$$

$$= 297{,}925.$$

Summations: Conclusion

- You need to know:
  - How to read, write & evaluate summation expressions like:

$$\sum_{i=j}^{k} a_i \qquad \sum_{i=j}^{\infty} a_i \qquad \sum_{x \in X} f(x) \qquad \sum_{P(x)} f(x)$$

  - Summation manipulation laws we covered.
  - Shortcut closed-form formulas, & how to use them.

Infinite Cardinalities

- Using what we learned about *functions* in §2.3, it's possible to formally define cardinality for infinite sets.
- We show that infinite sets come in different *sizes* of infinite!
- This also gives us some interesting proof examples.

Cardinality: Formal Definition

- For any two (possibly infinite) sets *A* and *B*, we say that *A* and *B* *have the same cardinality* (written $|A|=|B|$) iff there exists a bijection (bijective function) from *A* to *B*.
- When *A* and *B* are finite, it is easy to see that such a function exists iff *A* and *B* have the same number of elements $n \in \mathbf{N}$.

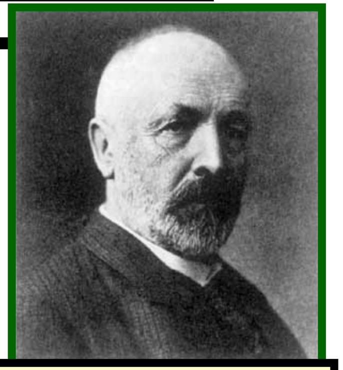
Countable versus Uncountable

- For any set *S*, if *S* is finite or if |*S*|=|**N**|, we say *S* is *countable*. Else, *S* is *uncountable*.
- Intuition behind "**countable**:" we can *enumerate* (generate in series) elements of *S* in such a way that *any* individual element of *S* will eventually be *counted* in the enumeration. Examples: **N**, **Z**.
- **Uncountable**: *No* series of elements of *S* (even an infinite series) can include all of *S*'s elements. Examples: **R**, $\mathbf{R}^2$, P(**N**)

Countable Sets: Examples

- **Theorem:** The set **Z** is countable.
  - **Proof:** Consider f:**Z**→**N** where $f(i)=2i$ for $i\geq 0$ and $f(i) = -2i-1$ for $i<0$. Note f is bijective.
- **Theorem:** The set of all ordered pairs of natural numbers (n,m) is countable.
  - Consider listing the pairs in order by their sum $s=n+m$, then by n. Every pair appears once in this series; the generating function is bijective.

Uncountable Sets: Example



Georg Cantor
1845-1918

- **Theorem:** The open interval $[0,1) :\equiv \{r\in\mathbf{R} \mid 0 \le r < 1\}$ is uncountable.
- **Proof** by *diagonalization*: (Cantor, 1891)
  - Assume there is a series $\{r_i\} = r_1, r_2, \ldots$ containing *all* elements $r\in[0,1)$.
  - Consider listing the elements of $\{r_i\}$ in decimal notation (although any base will do) in order of increasing index: ... *(continued on next slide)*

Uncountability of Reals, cont'd

A postulated enumeration of the reals:

$$r_1 = 0.d_{1,1}\,d_{1,2}\,d_{1,3}\,d_{1,4}\,d_{1,5}\,d_{1,6}\,d_{1,7}\,d_{1,8}\ldots$$
$$r_2 = 0.d_{2,1}\,d_{2,2}\,d_{2,3}\,d_{2,4}\,d_{2,5}\,d_{2,6}\,d_{2,7}\,d_{2,8}\ldots$$
$$r_3 = 0.d_{3,1}\,d_{3,2}\,d_{3,3}\,d_{3,4}\,d_{3,5}\,d_{3,6}\,d_{3,7}\,d_{3,8}\ldots$$
$$r_4 = 0.d_{4,1}\,d_{4,2}\,d_{4,3}\,d_{4,4}\,d_{4,5}\,d_{4,6}\,d_{4,7}\,d_{4,8}\ldots$$

- Now, consider a real number generated by taking all digits $d_{i,i}$ that lie along the *diagonal* in this figure and replacing them with *different* digits.

Uncountability of Reals, fin.

- *E.g.*, a postulated enumeration of the reals:
  $r_1 = 0.301948571\ldots$
  $r_2 = 0.103918481\ldots$
  $r_3 = 0.039194193\ldots$
  $r_4 = 0.918237461\ldots$
- OK, now let's add 1 to each of the diagonal digits (mod 10), that is changing 9's to 0.
- 0.4103… can't be on the list anywhere!

Countable vs. Uncountable

- You should:
  - Know how to define “same cardinality” in the case of infinite sets.
  - Know the definitions of *countable* and *uncountable*.
  - Know how to prove (at least in easy cases) that sets are either countable or uncountable.